

# Connections on higher order frame bundles

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**Abstract.** In the paper we present the analysis of connections on frame bundles of higher order contact, with special emphasis on the question of local flatness.

## 1. Introduction

The motivation for the research presented in this paper comes from the theory of continuous distributions of defects in continuous material bodies, in particular, the proposed generalization of the theory of continuous distributions of dislocations of simple elastic bodies to incorporate the higher order defects. [2, 3, 4] Recognizing that a definite G-structure can always be associated with the uniform elastic body the fundamental problem of this theory is the question of local integrability of such a structure. In purely mathematical terms this is equivalent to determining the existence of locally flat G-connections.

In this short paper we concentrate our efforts on studying the connections on frame bundles of order 2 and higher. We analyze both the form and the structure of these connections using as the fundamental concepts the notions of the fundamental form [6], the standard horizontal space of a frame [3] and, introduced here, the concept of the characteristic manifold of a connection. We discuss the conditions under which a connection on a bundle of frames of higher order becomes simple and locally flat. In this context we show the interplay between the simplicity, local integrability and the vanishing of the torsion. Although, our analysis, for most part, is presented in the general case of the semi-holonomic frame bundles, some interesting observations about the holonomic case are also made.

## 2. Canonical Forms

Let  $\mathcal{M}$  be an  $n$ -dimensional connected smooth manifold. Denote by  $\hat{H}^k(\mathcal{M})$  the space of all  $k$ -order frames of  $\mathcal{M}$ . Respectively, let  $\tilde{H}^k(\mathcal{M})$  be the space of all *holonomic*  $k$ -frames of  $\mathcal{M}$ . While  $\tilde{H}^k(\mathcal{M})$  is the space of  $k$ -order jets of all local diffeomorphisms of  $\mathbb{R}^n$  into  $\mathcal{M}$  with the source at the origin and the target anywhere in  $\mathcal{M}$ ,  $\hat{H}^k(\mathcal{M})$  can be thought of (recursively) as the space of first jets of all local sections of  $\hat{H}^{k-1}(\mathcal{M})$ .<sup>7</sup> For example, let  $f: U \rightarrow H^1(\mathcal{M})$  be a differentiable map of a neighborhood of the origin of  $\mathbb{R}^n$  into  $H^1(\mathcal{M})$  and such that  $\pi^1 \circ f: U \rightarrow \mathcal{M}$  is a local diffeomorphism where  $\pi^1: H^1(\mathcal{M}) \rightarrow \mathcal{M}$  is the standard projection. The first jet of  $f$  at 0 can be considered a (non-holonomic) 2-frame of  $\mathcal{M}$  at  $\pi^1(f(0))$ . If, in addition,  $f$  is such that the first jet of  $\pi^1 \circ f$  at 0 is equal to  $f(0)$  the corresponding 2-frame is called *semi-holonomic*. Extending this definition recursively to an arbitrary  $k$ -order we obtain the set of all semi-holonomic frames of  $\mathcal{M}$ , say  $H^k(\mathcal{M})$ . Hence, as mentioned in the Introduction, we shall be dealing only with semi-holonomic frames.

The space  $H^k(\mathcal{M})$  (also  $\tilde{H}^k(\mathcal{M})$ ) is a principal bundle over  $\mathcal{M}$ . Its structure group  $G^k$  is the fibre at 0 of  $H^k(\mathbb{R}^n)$ , i.e. the group of first jets at the origin of all local sections of  $H^{k-1}(\mathbb{R}^n)$  satisfying the condition of semi-holonomicity. The structure group of the bundle of holonomic frames  $\tilde{G}^k$  is the set of  $k$ -jets of all origin preserving local diffeomorphisms of  $\mathbb{R}^n$ . In particular,  $G^1 = \tilde{G}^1 = GL(n, \mathbb{R})$ . Given two, different order, frame bundles over  $\mathcal{M}$ , say  $H^k(\mathcal{M})$  and  $H^m(\mathcal{M})$ , where  $k > m$  there exists a natural projection  $\pi_m^k: H^k(\mathcal{M}) \rightarrow H^m(\mathcal{M})$  making  $H^k(\mathcal{M})$  in to an affine bundle over  $H^m(\mathcal{M})$  the structure group of which is the kernel  $N_m^k$  of the induced epimorphism  $\mu_m^k: G^k \rightarrow G^m$ . It is easy to see that  $N_m^k$  is the normal subgroup of  $G^k$ ,  $N_{k-1}^k$  is canonically isomorphic to the abelian vector group of all multilinear  $\mathbb{R}^n$ -valued  $k$ -forms on  $\mathbb{R}^n$  and that  $G^k$  is the semi-direct product of  $G^r$  and  $N_r^k$  for any  $r > k$ . Similarly,  $\tilde{H}^k(\mathcal{M})$ , which is a subbundle of  $H^k(\mathcal{M})$ , is an affine bundle over  $\tilde{H}^m(\mathcal{M})$ . Its structure group  $\tilde{N}_m^k$  contains symmetric multilinear  $\mathbb{R}^n$ -valued  $k$ -forms on  $\mathbb{R}^n$ .

To be able to introduce the notion of the *fundamental form* on a frame bundle let us recall [6, 7] first that given the (semi-holonomic)  $k$ -frame  $p^k$  there exists an isomorphism, called the *admissible isomorphism*,  $h^{k-1}: H^{k-1}(\mathbb{R}^n) \rightarrow H^{k-1}(\mathcal{M})$  such that  $p^k = j^1 h^{k-1}(e^{k-1})$  where  $e^{k-1}$  denotes the identity of the group  $G^{k-1}$ . Indeed, e.g., for any holonomic  $k$ -frame  $p^k$  there exists a local, about the origin of  $\mathbb{R}^n$ , diffeomorphism  $f: U \subset \mathbb{R}^n \rightarrow \mathcal{M}$  such that  $p^n = j^k f(0)$ . The corresponding isomorphism  $h^{k-1}$  is then defined by the condition that  $j^{k-1} f \circ f = h^{k-1} \circ j^{k-1} id$  where,  $j^{k-1} f: \mathcal{M} \rightarrow \tilde{H}^k(\mathcal{M})$ . The isomorphism  $h^{k-1}$  induces a linear isomorphism  $\tilde{h}^{k-1}: T_{e^{k-1}} H^{k-1}(\mathbb{R}^n) \rightarrow T_{\pi_{k-1}^k(p^k)} H^{k-1}(\mathcal{M})$ . Since  $H^{k-1}(\mathbb{R}^n) = \mathbb{R}^n \times G^{k-1}$  we have that  $T_{e^{k-1}} H(\mathbb{R}^n) = \mathbb{R}^n \oplus g^{k-1}$ . Here  $g^{k-1}$  is the Lie algebra of the structure group  $G^{k-1}$ . Gen-