CHAPTER 14

The S-Matrix Method

14.1. S-Matrix Formulation of the Basic Requirements of the Local Theory

A. THE CONCEPT OF EXTENDING THE S-MATRIX BEYOND THE MASS SHELL

The S-matrix method, put forward by Bogolubov, Medvedev and Polivanov is a special version of the LSZ theory. It is a development of an original idea to Heisenberg (1943) according to which, the content of relativistic quantum theory can be expounded in the language of the S-matrix. The S-matrix method arose in the process of generalizing Lagrangian quantum field theory (although there is no mention of the Lagrangian in it). It starts from the assumption that there is a more primitive object than the Heisenberg field and its T-products (which are the basic concepts of the LSZ formalism). The role of this fundamental object is fulfilled by the extension of the S-matrix beyond the mass shell on the basis of which the quantum fields and their T-products have already been constructed. In the S-matrix method, the derivation of the reduction formulae is considerably simplified. (The apparatus of formal variational derivatives makes the derivation of the various reduction formulae an automatic operation.) Another essential feature of the S-matrix method is that it is well suited to the treatment of the dynamical equations of quantum field theory (see, for example, [Z1]).

We turn to the exposition of the S-matrix method. First it is postulated that the scattering operator \( S \) is a unitary Poincaré-invariant operator in the Fock space \( \mathfrak{g} \) of a system of free relativistic particles (of type given in §7.3.C); for definiteness, \( \mathfrak{g} \) is identified with the Hilbert space of incoming particles. An irreducible system of (Wightman) free (or in-) fields \( \phi^{\text{in}}(\kappa)(x) \) acts in \( \mathfrak{g} \) with canonical commutation relations (under the normal connection between spin and statistics)

\[
[\phi^{\text{in}}_\ell(\kappa)(x), \phi^{\text{in}}_{\ell'}(\kappa')(y)]_\mp = \frac{1}{i} D_{\ell\ell'}^{(\kappa\kappa')}(x - y)
\]

and with two-point functions of type (13.52)

\[
\langle 0 | \phi^{\text{in}}_\ell(\kappa)(x) \phi^{\text{in}}_{\ell'}(\kappa')(y) | 0 \rangle = \frac{1}{i} D_{\ell\ell'}^{(\kappa\kappa')}(x - y).
\]

As always, the out-fields are related to the in-fields by the formula

\[
\phi^{\text{out}}(\kappa)(x) = S^* \phi^{\text{in}}(\kappa)(x) S.
\]
\[ D_{ll'}^{(\epsilon \kappa')} (z-y) \] is a Lorentz-covariant solution of the Klein-Gordon equation with mass \( m = m_\kappa > 0 \); it is represented in the form

\[ D_{ll'}^{(\epsilon \kappa')} (z-y) = Q_{ll'}^{(\epsilon \kappa')} \left( -i \frac{\partial}{\partial z} \right) D_m (z-y), \tag{14.4} \]

where \( Q_{ll'}^{(\epsilon \kappa')} \left( -i \frac{\partial}{\partial z} \right) \) is a Lorentz-covariant polynomial in \( \partial / \partial z \) with the properties *

\[ Q_{ll'}^{(\epsilon \kappa')} \left( -i \frac{\partial}{\partial z} \right) = (-1)^{F(\epsilon)} Q_{ll'}^{(\epsilon' \kappa)} \left( -i \frac{\partial}{\partial z} \right), \tag{14.5a} \]

(where \( F(\epsilon) = 0 \) or 1 depending on whether the field \( \phi_{\text{in}(\epsilon)} \) is bosonic or fermionic),

\[ Q_{ll'}^{(\epsilon \kappa')} \left( -i \frac{\partial}{\partial z} \right) = Q_{ll'}^{(\epsilon' \kappa)} \left( i \frac{\partial}{\partial z} \right). \tag{14.5b} \]

\( D_{ll'}^{(-\epsilon \kappa')} \) is the negative-frequency part of \( D_{ll'}^{(\epsilon \kappa')} \):

\[ D_{ll'}^{(-\epsilon \kappa')} (z-y) = Q_{ll'}^{(\epsilon \kappa')} \left( -i \frac{\partial}{\partial z} \right) D_m^{(-)} (z-y). \tag{14.6} \]

Similarly we define:

\[ D_{ll'}^{(\epsilon \kappa')} (x-y) = Q_{ll'}^{(\epsilon \kappa')} \left( -i \frac{\partial}{\partial z} \right) D_m^c (z-y), \tag{14.7} \]

\[ D_{ll'}^{(-\epsilon \kappa')} (x-y) = Q_{ll'}^{(\epsilon \kappa')} \left( -i \frac{\partial}{\partial z} \right) D_m^{\text{ret}} (z-y), \tag{14.8} \]

\[ D_{ll'}^{\text{adv}} (x-y) = Q_{ll'}^{(\epsilon \kappa')} \left( -i \frac{\partial}{\partial z} \right) D_m^{\text{adv}} (z-y). \tag{14.9} \]

We say that an extension of the S-matrix beyond the mass shell is defined if formal (left) variational derivatives of the S-matrix with respect to the in-fields

\[ \frac{\delta^n S}{\delta \phi_{\text{in}(\kappa_1)} (x_1) \ldots \delta \phi_{\text{in}(\kappa_n)} (x_n)} \equiv H(\kappa_1 \ldots \kappa_n | l_1 \ldots l_n (x_1, \ldots, x_n) \tag{14.10} \]

are defined and satisfy the conditions listed below. It is assumed that (14.10) is an operator-valued generalized function with respect to \( x_1, \ldots, x_n \) (with domain of definition in the Fock space \( \mathfrak{H} \)). The expression (14.10) is symmetric in its variables with respect to bosonic fields and antisymmetric with respect to fermionic fields; it is called a bosonic (or fermionic) operator if the sequence \( \phi_{\text{in}(\kappa_1)}, \ldots, \phi_{\text{in}(\kappa_n)} \) contains an even (or odd) number of fermionic fields.

Next, it is supposed that the Fourier transform of (14.10) admits a restriction to the mass shell with respect to the momentum \( p_j \) conjugate to any of the variables \( x_j \) (for example, by means of the procedure of restricting generalized functions to the mass shell described in §13.2.C). The definition of such a restriction is expressed in coordinate space by convolution with the commutator function \( D_{mj} (x_j) \). Thus we write the assumption of the existence of the restriction to the mass shell, say, with respect to \( p_1 \) in the form of the existence of the convolution

\[ \int D_{m_1} (x_1 - x'_1) \frac{\delta^n S}{\delta \phi_{\text{in}(\kappa_1)} (x'_1) \delta \phi_{\text{in}(\kappa_2)} (x_2) \ldots \delta \phi_{\text{in}(\kappa_n)} (x_n)} \, dx'. \tag{14.11} \]

* Formulae (14.5) guarantee the requisite property of (anti)symmetry of the (anti)commutator (14.1) and the property of the adjoint for the two-point function (14.2). (Concerning the explicit form of the covariants \( Q^{(\epsilon \kappa')}(p) \), see Appendix G, for example, the covariant expansion (G.4).)