The Wiener-Plancherel Theorem on $\mathbb{R}^n$

Ward R. Evans

The MITRE Corporation
7525 Colshire Rd.
McLean, VA 22102
USA

ABSTRACT

This paper will discuss the recent proof of the $n$-dimensional Wiener-Plancherel formula [2] and some of the problems encountered in extending Wiener's Generalized Harmonic Analysis (GHA) to $\mathbb{R}^n$. A key feature of this theory is the definition of an appropriate $s$-function, which in turn depends on the convergence criteria selected. This paper considers convergence over rectangles.

1 Introduction

Norbert Wiener devised GHA to solve problems that could not be solved using classical techniques of harmonic analysis. In particular Wiener wanted a theory "adequate for the treatment of a ray of white light which is supposed to endure for an indefinite time" [5]. In modern terms, one would say that Wiener wanted a harmonic analysis of bounded functions. In [5], Wiener successfully solved this problem for functions bounded on $\mathbb{R}$.

A central theorem in GHA is the Wiener-Plancherel formula, an analogue of the Plancherel formula. It applies to functions having bounded quadratic means. Since the Wiener-Plancherel formula is a statement about the equality of means, one should not be surprised to find that the extension of the formula to $\mathbb{R}^n$ depends on convergence criteria determined by geometrical constraints. The generalization discussed in this paper depends on the local behavior of the function on $n$-dimensional rectangles.

While the work described in this paper is contained in more detail in [2], it seems useful to have a shorter exposition of the theory. I would like to thank my coauthors of [2], John Benedetto and George Benke, for their help, encouragement, and friendship.

2 The Wiener-Plancherel Formula on $\mathbb{R}$

In this section we shall examine the classical Wiener-Plancherel formula on $\mathbb{R}$, briefly discuss its place in Generalized Harmonic Analysis (GHA), and define terms and spaces needed for the extension to $\mathbb{R}^n$. 

Definition 2.1
a. The space $BQM (\mathbb{R}^n)$ of functions having bounded quadratic means is the set of all functions
$$\varphi \in L^1_{\text{loc}} (\mathbb{R}^n) = \left\{ f : f \text{ measurable on } \mathbb{R}^n \text{ and } \int_K |f| < \infty \text{ } \forall \text{ compact } K \subset \mathbb{R}^n \right\},$$
where
$$B (\varphi) \equiv \sup_{T > 0} \frac{1}{|R_T|} \int_{R_T} |\varphi (t)|^2 \, dt < \infty,$$
$$T = (T_1, \ldots, T_n) \in \mathbb{R}^n, T > 0 \text{ means } T_i > 0 \text{ for } i = 1, \ldots, n, \text{ and } |R_T| \text{ denotes the Lebesgue measure of } R_T = \{ t \in \mathbb{R}^n : |t_j| \leq T_j \text{ for } j = 1, \ldots, n \}.$$
b. The Wiener space, $W(\mathbb{R}^n)$, is the set of all functions $\varphi \in L^1_{\text{loc}} (\mathbb{R}^n)$ for which
$$W (\varphi) \equiv \int_{\mathbb{R}^n} \frac{|\varphi (t)|^2}{(1 + t_1^2) \cdots (1 + t_n^2)} \, dt < \infty.$$

Wiener [3] proved the first inclusion of the following theorem for $n = 1$; the theorem as stated is proved in [2].

Theorem 2.2 $BQM (\mathbb{R}^n) \subseteq W (\mathbb{R}^n) \subseteq S' (\mathbb{R}^n)$, where $S' (\mathbb{R}^n)$ is the class of tempered distributions on $\mathbb{R}^n$.

A concept crucial to GHA is the concept of the $s$ function. In this section we shall give the definition of the $s$ function for $n = 1$; we shall extend the definition to n-dimensions in the next section.

Definition 2.3 Let $\varphi \in L^1_{\text{loc}} (\mathbb{R})$ and define
$$s (\gamma) = \int_{-\infty}^{\infty} e (t, \gamma) \varphi (t) \, dt, \gamma \in \mathbb{R}$$
where
$$e (t, \gamma) = \begin{cases} \exp(-2\pi i t) \gamma - 1, & |t| \leq 1 \\ \exp(-2\pi i t), & |t| > 1. \end{cases}$$

Theorem 2.4 If $\varphi \in W (\mathbb{R})$ then
a. $s \in L^2_{\text{loc}} (\mathbb{R})$
b. $\Delta s \in L^2 (\mathbb{R})$, where $\Delta s (\gamma) = s (\gamma + \lambda) - s (\gamma - \lambda)$.

Proof:
a. Write $s (\gamma) = s_1 (\gamma) + s_2 (\gamma)$ where
$$s_1 (\gamma) = \int_{-1}^{1} \varphi (t) \frac{e^{-2\pi i t \gamma} - 1}{-2\pi i t} \, dt,$$
$$s_2 (\gamma) = \int_{|t|>1} \varphi (t) \frac{e^{-2\pi i t \gamma}}{-2\pi i t} \, dt.$$