Chapter 23: Proof and Proving

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ABSTRACT

Proof is an essential characteristic of mathematics and as such should be a key component in mathematics education. Translating this statement into classroom practice is not a simple matter, however, because there have been and remain differing and constantly developing views on the nature and role of proof and on the norms to which it should adhere.

Different views of proof were vigorously asserted in the reassessment of the foundations of mathematics and the nature of mathematical truth which took place in the nineteenth century and at the beginning of the twentieth, a reassessment which gave rise to well-known and widely divergent philosophical stands such as logicism, formalism and intuitionism. These differences have now been joined by disagreements over the implications for proof of ‘experimental mathematics’, ‘semi-rigorous mathematics’ and ‘almost certain proofs’, concepts and practices which have emerged on the heels of the enormous growth of mathematics in the last fifty years and the ever-increasing use of computers in mathematical research. If these and earlier controversies are to be reflected usefully in the classroom, mathematics educators will have to acknowledge and become familiar with the complex setting in which mathematical proof is embedded. This chapter aims at providing an introduction to this setting and its implications for teaching.

It is not merely as a reflection of mathematical practice that proof plays a role in mathematics education, however. Proof in its full range of manifestations is also an essential tool for promoting mathematical understanding in the classroom, however artificial and unnatural its use there may seem to the beginner. To promote understanding, however, some types of proof and some ways of using proof are better than others. Thus this chapter also aims at providing an introduction to didactical issues that arise in the use of proof.

The chapter first discusses the great importance accorded in mathematical practice to the communication of understanding, pointing out the place of proof in this endeavour and the implications for mathematics teaching. It then identifies and assesses some recent challenges to the status of proof in mathematics from mathematicians and others, including predictions of the ‘death of proof’. It also examines and largely seeks to refute a number of challenges to the importance of proof in the curriculum that have arisen within the field.
of mathematics education itself, sometimes prompted by external social and philosophical influences.

This chapter continues by looking at mathematical proof, and the mathematical theories of which it is a part, in terms of their role in the empirical sciences. There are important insights into the use of proof in the classroom that may be garnered through a deeper understanding of the mechanism by which mathematicians, nominally practitioners of a non-empirical science, make an indispensable contribution to the understanding of external reality.

Later sections examine the use of proof in the classroom from various points of view, proceeding from the premise that one of the key tasks of mathematics educators at all levels is to enhance the role of proof in teaching. The chapter first reports upon some ambivalent but nevertheless encouraging signs of a strengthened role for proof in the curriculum, and turns to a discussion of proof in teaching, offering a model defining its full range of potential functions. The important distinction between proofs which prove and proofs which explain is then introduced, and its application is presented at some length with the help of examples.

1. PROOF AND UNDERSTANDING

The most significant potential contribution of proof in mathematics education is the communication of mathematical understanding. One comes to appreciate the importance of this seemingly trite determination if one examines critically the view of proof adopted by the ‘new math’ movement of the 1950’s and 1960’s.

The belief implicit in the ‘new math’ was that the secondary-school mathematics curriculum better reflects mathematics when it stresses formal logic and rigorous proof. This belief rested upon two key assumptions:

a) that in modern mathematical theory there are generally accepted criteria for the validity of a mathematical proof; and

b) that rigorous proof is the hallmark of modern mathematical practice.

Both of these beliefs can be seen to be false (Hanna, 1983). First of all, even a cursory revisiting of the major accounts of the nature of mathematics (logicism, formalism, intuitionism and quasi-empiricism) makes it obvious that these significant schools of mathematical thought hold widely differing views on the role of proof in mathematics and on the criteria for the validity of a mathematical proof.

Second, an examination of mathematical practice shows clearly that in the eyes of practising mathematicians rigour is secondary in importance to understanding and significance, and that a proof actually becomes legitimate and convincing to a mathematician only when it leads to real mathematical under-