CHAPTER 11

THE DETERMINANTS OF TOEPLITZ
AND HANKEL RANDOM MATRICES

In this chapter, the perturbation and orthogonalization methods are applied to the analysis of determinant distribution of Toeplitz and Hankel matrices. Besides, limit theorems of the law of large numbers and the central limit theorem type are proved: the general form of limiting distributions for determinants of random Toeplitz and Hankel matrices has been found.

§1 Limit Theorem of the Law of Large Numbers Type

By the Hankel random matrix of order \( n \) we mean the matrix \( \Gamma_n = (\xi_i + j)^{i,j=1,r} \) (where \( \xi_i, i = 2, 2n \) are random complex variables), as well as the matrix of the kind \( (u_n(w, \xi_i + \xi_j)) \) (where \( \xi_i \) are random variables, and \( u_n(w, t) \) is a random function such that \( u_n(w, \xi_i + \xi_j) \) are random variables).

The square matrix \((\xi_{i-j})_{i,j=0}^{n-1}\) is called a random Toeplitz matrix of order \( n \), where \( \xi_i, i = 0, \pm 1, \ldots, \pm (n-1) \) are complex variables. The Toeplitz matrix will be Hermitian if it satisfies the conditions \( \xi_{i-p} = \overline{\xi_p}, p = 0, n - 1 \).

It is possible for the determinants of such matrices to formulate limit theorems of the law of large numbers type. Let there exist \( E \ln^2 |\det \Gamma_n| \) and \( \gamma_k = E[\ln |\det \Gamma_n|/\sigma_k^{(n)}] - E[\ln |\det \Gamma_n|/\sigma_k] \), where \( \sigma_k^{(n)} \) are the smallest \( \sigma \)-algebras, with respect to which the random variables \( \xi_i, l = k + 1, 2n \) are measurable. If for a certain sequence of constants \( c_n \),

\[
\lim_{n \to \infty} c_n^{2} \sum_{k=1}^{n} E\gamma_k^2 = 0, \tag{11.1.1}
\]

then \( \text{plim}_{n \to \infty} c_n^{-1}(\ln |\det \Gamma_n| - E \ln |\det \Gamma_n|) = 0 \).

If \( E\gamma_k^2 \) does not exist, then it is possible to use the conditions of Corollary 5.4.1.

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Theorem 11.1.1. Let $\Gamma_n = (u(\xi_i + \xi_j)), u(x)$ is a nonrandom real even Borel function, for each $n$ the random variables $\xi_i^{(n)}$ are independent

$$\lim_{n \to \infty} c_n^{-2} \sum_{k=1}^{n} \mathbb{E} \left[ \frac{\ln(1 + \delta_n)}{\sigma_k^{(n)}} \right] - \mathbb{E} \left[ \ln \frac{1 + \delta_n}{\sigma_k^{(n)}} \right]^2 = 0, \quad (11.1.2)$$

where $c_n$ is some sequence of constant values, $\sigma_k^{(n)}$ is the smallest $\sigma$-algebra with respect to which the random variables $\xi_l$, $l = k + 1, n$ are measurable, 

$$\delta_n = iu(0) + (R_k \overline{u_k}, u_k), R_k = (I + i\Gamma_n)^{-1}.$$ 

and $\Gamma_n^{k}$ is obtained from the matrix $\Gamma_n$ by replacing the entries of the $k$th vector row and of the $k$th vector column by zeros.

Then 

$$\operatorname{plim}_{n \to \infty} c_n^{-1} \{ \ln \det(I + \Gamma_n^{2}) - \mathbb{E} \ln \det(I + \Gamma_n^{2}) \} = 0.$$ 

Proof. Obviously, the matrix $\Gamma_n$ is symmetric and 

$$\ln \det(I + \Gamma_n^{2}) = 2|\ln \det(I + i\Gamma_n)|.$$ 

Condition (11.1.2) is obtained by using the formula 

$$\ln \det(I + i\Gamma_n) - \ln \det(I + i\Gamma_n^{k}) = \ln(I + \delta_n).$$ 

Theorem 11.1.1 is proved.

Note that (11.1.2) holds under $\sup_n \sup_{k \leq \infty} |1 + \delta_n| = c, \lim_{n \to \infty} c_n^{2} n = 0$.

As has been shown in Corollary 5.4.1, it is possible to demand the existence not of $\mathbb{E} \ln^2 \det(I + \Gamma_n^{2})$ but of $\mathbb{E} \ln^{1+\varepsilon} \det(I + \Gamma_n^{2}), \varepsilon > 0$. The condition of Theorem 11.1.1 can be weakened if we use limit random theorems for normalized spectral functions (see Chapter 9).

Theorem 11.1.2. Let $\Gamma_n = (u(\xi_i + \xi_j)), u(x)$ be a nonrandom real even Borel function, the random variables $\xi_i^{(n)}, i = \overline{1, n}$ for each $n$ be independent and given on a common probability space, and $\mu_n(x)$ be a normalized spectral function of the matrix $\Gamma_n$.

Then with probability 1 for almost all $x$, $\lim_{n \to \infty} [\mu_n(x) - F_n(x)] = 0$, where $F_n(x)$ is a nonrandom distribution function, whose Stieltjes transformation is 

$$\int (1 + itx)^{-1} dF_n(x) = n^{-1} \mathbb{E} \operatorname{Tr}(1 + it\Gamma_n)^{-1}.$$ 

Let us now study Hankel matrices of the kind $\Gamma_n = (\xi_i + j)$. Let $\theta_k = (\delta_{i,k-j}), i, j = \overline{1, n}$, where $\delta_{ij}$ is the Kronecker symbol.