SMOOTH PARAMETRIC SURFACES AND n-SIDED PATCHES

JOHN A. GREGORY, VINCENT K.H. LAU, JIANWEI ZHOU
Department of Mathematics and Statistics
Brunel University, Uxbridge, UB8 3PH,
England.

ABSTRACT. The theory of 'geometric continuity' within the subject of CAGD is reviewed. In particular, we are concerned with how parametric surface patches for CAGD can be pieced together to form a smooth $C^k$ surface. The theory is applied to the problem of filling an n-sided hole occurring within a smooth rectangular patch complex. A number of solutions to this problem are surveyed.

1. Introduction

The topics addressed by these tutorial lectures are those of polygonal patches and the theory of geometrically smooth parametric surfaces. In particular, we will consider the problem of filling a polygonal hole occurring within a smooth rectangular patch complex. This problem, which is frequently encountered by current 'free-form' or 'sculptured' surface modellers, illustrates the need for the theory of 'geometric continuity' within the subject of CAGD (Computer Aided Geometric Design). This theory is concerned with how parametric surface patches can be pieced together to give a smooth $C^k$ surface.

A $C^k$ surface is one which locally admits a $C^k$ parameterization. Thus the surface can be considered as a collection of overlapping patches, each defined as a $C^k$ map from an open domain in $\mathbb{R}^2$ into $\mathbb{R}^3$. In CAGD, however, the surface is composed of a number of non-overlapping patches, each defined on a closed domain in $\mathbb{R}^2$. We are thus concerned with how to join together such closed patches in order to obtain a $C^k$ surface.

The simplest type of join between two patches is that of a 'parametric continuous' $C^k$ join, where the parameter domains of two patches abut along a common edge and the surface is $C^k$ on the composite domain. In effect, the surface is one composite patch, the domain being the union of the individual patch domains. This is the situation most frequently encountered when composing a surface of rectangular patches, where the patch complex can be considered as a mapping from a parametric domain subdivided by a regular rectangular mesh. More generally, two patches can have a 'geometric continuous' $G^k$ join. Here we will see that, locally, there is a $C^k$ reparameterization in which the composite surface is parametrically $C^k$. A parametric continuous join is then the...
special case of where the reparameterization is defined by the identity map.

Whilst current surface modellers use almost entirely rectangular patches, it is impossible to model a complex surface as a single map from a regular rectangular meshed domain. More complex surface topologies will require n-sided polygonal holes within the rectangular patch complex to be filled in. This situation typically arises where two or more rectangularly meshed surfaces are to be blended together. One solution might be to allow n rectangular patches to meet at a common vertex in $\mathbb{R}^3$, whilst another solution might be the construction of a special polygonal patch. In either case the concept of a geometrically continuous join of the patches is necessary.

The lectures are organised as follows. Section 2 develops the geometric continuity tools needed for the study of smooth parametric surface patch complexes. In section 3, rectangular patch representations are briefly reviewed and the n-sided polygonal hole problem is described. Solutions to the polygonal hole problem are then discussed in sections 4 and 5.

2. Geometrically Smooth Parametric Surfaces

The geometric continuous $G^k$ join of two surface patches is a terminology that is now well established in CAGD. However, the differential geometer might well prefer to state that two such patches meet with 'contact of order $k$'. The idea will be explained through that of a reparameterization of the surface, see, for example, DeRose [21], Höllig [52], Gregory and Hahn [39], and Hahn [48]. Here we will summarize some of the material contained in Gregory [37] which is based on the exposition of Hahn [48]. We first introduce the reparameterization approach by considering the simpler case of planar curves.

2.1 Geometric continuous curves

Consider the two planar curve segments

$$p(t) = (t, t^2 + t^3), \ -2 \leq t \leq 0,$$
$$q(t) = (2t, 4t^2), \ 0 \leq t \leq 1.$$ (2.1)

Clearly $p(2t), -1 \leq t \leq 0$, describes the same curve segment as $p(t)$ but with different parameterization. Moreover, $p(2t)$ and $q(t)$ are (parametrically) $C^2$ at $t = 0$. We thus say that the original parameterizations $p(t)$ and $q(t)$ have a geometric continuous $G^2$ join. This idea will be formalized in the definition below but first the need for the use of 'regular' parametric representations is explained.

A univariate parameterization $p : [a,b] \to \mathbb{R}^2$ is said to be a regular parametric representation of class $C^k$, $k \geq 1$, if the first derivative tangent vector $p^{(1)}(t)$ does not vanish and if the component functions of $p$ are $k$ times continuously differentiable on $[a,b]$. Regularity, that is