ABSTRACT. A model for a dense packing of disks rolling on each other is presented. This model might have application for mechanical gearworks, for turbulence or for tectonic motion. A full classification of solutions with fourfold loops is given. The fractal dimensions are calculated and compared to Kolmogoroff scaling.

1. Introduction

Is it possible to tile a plane with wheels rolling on each other such that all the area is covered with wheels? This rather exotic question can arise in various contexts. One could imagine the wheels to be eddies on the surface of an incompressible fluid and then ask if the fluid motion can be totally decomposed into stable eddies. Or, one could think of mechanical roller bearings between two moving surfaces, like two tectonic plates, and then ask if one can completely fill the space between the rolling cylinders with other rolling cylinders such that no cylinder exerts any frictional work on another one. The question we are asking is, in fact, geometrical.

The original motivation for studying this problem was the enigmatic observation that over very extended areas, called "seismic gaps," two tectonic plates can creep on each other without producing neither earthquakes nor the amount of heat expected from usual friction forces. One such region is a part of the San Andreas Fault that extends over more than one hundred kilometers between Los Angeles and San Francisco.

As a possible mechanism to explain this behavior one could think that the material between the plates, which is called "gouge," organizes itself in such a way that it acts like a bearing. Since in a bearing one has no gliding friction but only rolling friction this could explain the lack of measurable heat production. On the other hand, in roller-skates for instance, mechanical bearings work because the individual balls or cylinders of the bearing are kept separate from each other by leaving rather big empty spaces between them. Under the pressure of kilobars that push tectonic plates against each other such empty spaces cannot exist. The space between the rolling cylinders must be filled with rolling matter such that the motion of the main cylinders is not hindered by gliding friction. So the main cylinders should roll on secondary cylinders which themselves roll on successive generations of smaller and smaller cylinders as shown in Fig. 1. Evidently provided such a
bearing exists it can only be constructed iteratively and will therefore probably be self-similar\(^\dagger\). Several seismic measurements have in fact indicated self-similar or turbulent motion within the gouge.\(^2\)

![Schematic two-dimensional cut through a roller bearing between two tectonic plates. The inserts show how the holes could be filled with rotating cylinders.](image)

**Fig. 1:** Schematic two-dimensional cut through a roller bearing between two tectonic plates. The inserts show how the holes could be filled with rotating cylinders.

In order for the above model to work in practice various conditions still need to be fulfilled: The individual stones within the gouge must be round; there must be a dynamics under which the system naturally evolves into the bearing; at some lower cut-off which can be given by the roughness of the surface of the stones another mechanism must take over and finally one has to justify considering cylinders or two-dimensional cuts instead of the full three-dimensional motion. It is not the aim of this course to deal with these questions. We want to concentrate just on the existence and construction of the space-filling self-similar bearings and study their geometrical properties.

Tiling space with circles by putting iteratively in each hole between three circles the circle that tangentially touches all three (see Fig. 2) is an old problem often known under the name of “Apollonian packing.” It dates back to Apollonius of Perga who lived around 200 B.C. and much work has been done since as briefly presented for instance in Mandelbrot’s book.\(^3\) The space left over between circles is a fractal but despite much effort it has not yet been possible to determine the value of the fractal dimension analytically. The best numerical estimate is \(d_f \approx 1.3058\).\(^4\)

The best way to construct Apollonian packings is by using iteratively circle-conserving mappings (Möbius transformations).\(^5\) This technique turns out to be also suited to our problem of bearings. We will in the following section describe

\(\dagger\) We will not distinguish in this course between self-similar and self-inverse.\(^3\)