NUMERICAL SIMULATION OF RICHARDS EQUATION: CURRENT APPROACHES AND AN ALTERNATE PERSPECTIVE

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ABSTRACT. The transient flow of water in saturated-unsaturated media is described by a nonlinear parabolic partial differential equation, familiarly known as Richards equation. Numerical solution of Richards equation is often beset with difficulties related to stability, convergence, and verification, particularly when water saturations are low and when material heterogeneities exist. It is suggested that these difficulties arise largely due to the fact that conventional numerical technical techniques based on finite differences and finite elements do not take into account the nature of the local flow geometry in estimating fluxes. Nor do they recognize that the Darcy-Buckingham equation, in the presence of gravity, heterogeneities or nonuniform flow geometry, is an implicit statement relating flux to the potential distribution between two surfaces of equal potential. Moreover, for an elemental volume in a transient nonlinear system, capacitance has to be defined in an operational sense, being specifically associated with a chosen location of observation within the elemental volume. Finally, in order to compute fluxes accurately, the time-averaging factor has to be made a function of space and of time. Theoretical discussions are provided to demonstrate how these ideas may be synthesized to solve the problem of transient flow in a flow tube of non uniform cross sectional area.

1. INTRODUCTION

1.1. Motivation

The transient flow of water in an isothermal porous medium under conditions of partial saturation is often expressed in the form of a partial differential equation. Originally proposed by L.A. Richards in 1931, this governing equation is subject to the important assumption that the air phase is at a constant pressure within the zone of partial saturation. Richards equation is extremely non-linear in nature due to the strong dependences of material properties on the dependent variable, water phase pressure. As a consequence, closed form solutions to Richards equation are extremely difficult to obtain, especially when one is interested in multidimensional heterogeneous systems with complex geometries. Therefore, for applying Richards equation to any realistic field problem, the preferred approach among researchers at the present time is the use of numerical models.

H. J. Morel-Seytoux (ed.), Unsaturated Flow in Hydrologic Modeling
Within the past thirty years a variety of numerical models have appeared in the literature for solving Richards equation (e.g. Brutsaert, 1971; Cooley, 1971; Freeze, 1971; Narasimhan and Witherspoon, 1978; Neuman, 1973; Rubin et al., 1964; and many others). Despite the availability of many such algorithms, practical difficulties do exist in the credible implementation of these models. These difficulties relate not only to the task of merely obtaining a solution (stability; convergence) but also to the verification of the solutions that are so obtained.

The present work is motivated by a desire to identify the causes of these difficulties and to explore rational ways of overcoming them.

1.2. Scope

The transient transport process in the vadose zone is one that involves multiple fluid phases and heat. Yet, Richards equation idealizes the system purely in terms of single phase water transport. Some researchers (Morel-Seytoux, 1987) have attempted to minimize the effect of this constraint by treating the vadose zone as a two-fluid system involving water and air or as a multi-component system involving heat as well (Philip and de Vries, 1957; Sophocleos, 1979). In the present work we will not be concerned with these more general approaches and we shall restrict ourselves to the single-phase isothermal idealization of Richards equation.

It has been recognized in the literature that the strong non-linearity of Richards equation could be eased by simply casting the equations using water content rather than pressure head as the dependent variable. Because, in heterogeneous media water content is discontinuous at material interfaces, this formulation has to be supplemented by continuity criteria on capillary pressure head when applied to heterogeneous media. Thus, the ultimate solution of Richards’ equation has to take into account the variation of fluid pressure. Because most realistic field problems in the earth sciences involve heterogeneous media, we will devote our attention in the present work exclusively to the pressure-head based formulation.

A majority of the numerical models proposed for solving Richards equation involve the discretization of the flow domain as well as time into finite subsets and integrating the equation in terms of discrete sums. These methods fall into two general categories, the Integral Finite Difference Methods IFDM (*) and the Finite Element Methods (FEM). A relatively new technique known as the Boundary Element Method (BEM) is used by some researchers to solve the Richards equation. This method consists generally in discretizing the boundary surface of each material within the system and numerically integrating the Green’s functions over these segments. However, the Green’s Functions are primarily well-suited for linear problems and are not well-defined for non-linear equations. In the present work we shall restrict ourselves to the IFDM and the FEM.

The scope of this work is a modest one of recalling certain well-accepted numerical modeling approaches and to look for rational ways of extending beyond these approaches. This work does not include a detailed review of all the relevant literature on numerically modeling Richards’ equation.

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We include the classical Finite Difference Methods (FDM) as subsets of the IFDM in the present work.