The n-Person War of Attrition

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1. Introduction

The War of Attrition (WA) was one of the earliest examples studied in the use of the theory of games to understand animal behavior (see Maynard Smith (1974)). The setup is that two contestants compete for a prize worth \( V(V > 0) \), and the one who is prepared to wait longer collects the prize; both contestants incur a cost equal to the length of time taken to resolve the contest. Symbolically, if \( E(x,y) \) denotes the amount gained by a contestant prepared to wait time \( x \) when the opponent is prepared to wait \( y \),

\[
E(x,y) = \begin{cases} 
V-y & \text{if } x > y \\
-z & \text{if } x < y
\end{cases}
\]

(1)

with

\[
E(x,x) = \frac{V}{2} - z \quad x \in [0,\infty) \quad y \in [0,\infty) .
\]

Such a game has precisely one evolutionarily stable strategy or ESS (Bishop and Cannings (1976)), i.e. a strategy such that if played by a population, no mutant using another strategy can invade. This ESS is to wait for a time \( z \) drawn at random from the exponential distribution with mean \( V \), i.e. density \( \frac{1}{V} \exp(-z/V) \) (\( z > 0 \)).
Several modifications and generalisations of the basic model (1) have been investigated (e.g. Bishop and Cannings (1978), (1986), Haigh and Rose (1980), Maynard Smith and Parker (1976)). Here we analyze several different models for the generalisation of (1) to $n$-person conflicts. The common feature of the models is that the contestants pay a cost, measured as the length of time they participate in the contest, and are prepared to do so because of their hope of receiving a reward. An ESS for an $n$-person WA (if one exists) will be a probability distribution $G$, that governs the length of time a contestant is prepared to wait, with the property that, if all the population are using $G$, then no invader who uses a different distribution $H$ can establish himself from a small base, i.e. a small fraction $\varepsilon$ of the population.

The various models described below hopefully reflect some of the situations which might arise in nature. There are many situations in which a number of individuals ($n > 2$) are simultaneously in conflict; a pack with a kill must establish the order in which individuals may feed (though a prior "pecking order" may exist); territorial animals will need to establish how a set of potential territories are allocated; male bees may compete for the right to mate with the queen. These examples suffice to indicate that there are occasions in which $n$ individuals compete, and that there may be only a single reward available, or a number of rewards, and provide the motivation for the choice of models below.

The only prior work of which we are aware is some discussion by Palm (1984) of the general notion of an ESS with $n$ contestants, but no specific models are considered.

2. The Models

Model A. With $n$ players competing for one reward, value $V$, each player independently selects a time he is prepared to wait, in the hope of outlasting the $(n-1)$ opponents. Once chosen, this time is fixed: as some players drop out, those remaining are not allowed to alter their initial "bids".

Here player $i$ will select a random time $X_i$ drawn from some distribution $G$ that depends only on $V$ and $n$. The reward to player 1 can be written

$$E(X_1;X_2,\ldots,X_n) = \begin{cases} V - W_1 & \text{if } X_1 > W_1 \\ -X_1 & \text{if } X_1 < W_1 \end{cases}$$

(2)

where $W_1 = \max(X_2,X_3,\ldots,X_n)$, and the reward to the other players follow in a symmetric manner. We can safely ignore consideration of the case $X_1 = W_1$ since this has probability zero of arising when the distribution of $\{X_1\}$ is continuous; and this is the