CHAPTER X

Some Additional Remarks

10.1. For a curve in an $n$-dimensional Euclidean space $E^n$ which obeys certain conditions of regularity, to each point of the curve there is put in correspondence $(n-1)$ numbers of $k_1, k_2, \ldots, k_{n-1}$ of the curvatures at the point of the curve. Let us set

$$\kappa_i(s) = \int_0^s k_i(\sigma) \, d\sigma$$

where integration is carried out with respect to the arc length. The functions obtained in this way will be called integral curvatures of the curve. There arises a problem - to give such a definition of the $i$th integral curvature which could be applied to an essentially wider class of curves than the class considered in differential geometry. Accordingly, the contents of Chapter V can be considered as a theory of the curves of a limited integral first curvature. Chapter IV presents a theory of curves of a limited integral first curvature in a space $S^n$. Theories of curves of a generalized integral curvature of the $r$ order are considered in the works by I.P. Majnick [13 - 18]. The class $H_r$ of curves of a finite integral curvature is defined in the following way. First we introduce the notion of an $r$-dimensional osculating plane as a limit of oriented secant planes. Let us assume that $X_0 < X_1 < \cdots < X_r$ are the points of the curve $K$ not lying in one $(r-1)$-dimensional plane. The symbol $P(X_0, X_1, \ldots, X_r)$ will denote an $r$-dimensional plane $P$ passing through the given points $X_i$ and oriented in such a way that a sequence of vectors $(X_0X_1, X_0X_2, \ldots, X_0X_r)$ is a right-hand reference frame in the plane $P$. Let $X$ be a point of the curve $K$. The limit of the planes $P(X_1, X_2, \ldots, X_r, X)$, where $X_1 < X_2 < \cdots < X_r < X$ when $X_1 \to X, \ldots, X_r \to X$, if it exists, is termed a left-hand osculating plane at the point $X$ of the curve $K$. Analogously, the limit of the secant planes $P(X, X_1, \ldots, X_r)$, when $X_1 \to X, \ldots, X_r \to X$ from the right, is termed a right-hand osculating plane at the point $X$. If the left- and the right-hand osculating $r$-dimensional planes at the point $X$ of the curve $K$ are different, then at this point we can also define intermediate $r$-dimensional osculating planes, which can be sufficiently numerous (but not greater than $(r-1)$). The possible variations of constructing a set of intermediate $r$-dimensional planes are not going to be a subject of our discussion.
Let us assume that the curve $K$ is such that none of its arcs lies in one $(r - 1)$-dimensional plane and at each its point $X$ it has a left-hand (if this point is not the beginning of the curve) and a right-hand (if $X$ is not the end of $K$) $r$-dimensional osculating planes. A set of all osculating $r$-dimensional planes of the curve $K$ is ordered in a natural way. The curve $K$ belongs to the class $H_r$ if the quantity

$$\kappa_r(K) = \sup \sum_{i=0}^{m-1} (P_i, P_{i+1})$$

is finite. Here $P_0 < P_1 < \cdots < P_m$ is an arbitrary finite sequence of osculating $r$-dimensional planes of the curve $K$, and the upper boundary is chosen with respect to a set of all such sequences. The quantity $\kappa_r(K)$ is termed an $r$-th integral mean curvature of the curve $K$.

As is shown in [14], if the curve $K$ belongs to the class $H_r$, then it also belongs to the class $H_s$, for any integer $s$, where $1 \leq s \leq r$. For the curves obeying the conventional conditions of regularity in differential geometry, $\kappa_r(K)$ coincides with the integral from a curvature of order $r$ with respect to the arc length. The problem on the possibility of defining an integral curvature of order $r$ and the class of curves $H_r(K)$ by way of approximating a curve by polygonal lines remains unsolved.

10.2. Theorems on natural parametrization in Chapter IX are proved in a way that is closest to that used in the textbooks on differential geometry (instead of the term 'natural parametrization' they commonly use the term 'natural equation'). Here another approach is possible, associated with the use of the notion of a parallel displacement in an arbitrary fibre-bundled space. This approach is also of interest since it contains a certain method of introducing the curves of a limited integral curvature of order $r$, different from that mentioned in 10.1.

Let $K$ be a curve on the sphere $S^n$ in the space $E^{n+1}$. Let us place the space $E^n$ in the form of a certain hyperplane into $E^{n+1}$. Let us assume that the sphere $S^n$ is rolling over the plane $E^n$, touching $E^n$ at every moment of time $t$ at the points of the curve $K$. Formally this means that the sphere $S^n$ moves in $E^{n+1}$ in such a way that its instantaneous axis of rotation is parallel to $E^{n+1}$, and the velocity at the point of touching of $S^n$ and $E^n$ equals zero. In this case the point of touching of $S^n$ and $E^n$ draws in $E^n$ a certain curve $Q$ which is called a development of the curve $K$. It can be easily proved that if the curve $K$ is regular, then at every $i = 1, 2, \ldots, n - 1$ the curvatures $k_i$ at the corresponding points of the curves $K$ and $Q$ have equal values and hence the natural equations of the curve and its developments coincide.

Let $K$ be a curve of a finite complete torsion in $E^3$, $(s(u), \kappa(u), \tau(u))$, $a \leq u \leq b$, be its natural parametrization. In this case $(\kappa(u), \tau(u))$, $a \leq u \leq b$, is a natural parametrization of the indicatrix of the tangents $Q$ of the curve $K$. At the same time, it is a natural parametrization of the development $R$ of