Amphiphilic membranes are characterized by a very small value of their surface tension. As a consequence, as it was pointed out long ago by Helfrich\textsuperscript{1}, the free energy associated with their configuration is essentially determined by curvature. This also implies that shape fluctuations (undulations) are much more violent in these membranes than they are in usual interfaces. I shall discuss in this lecture the effects of such fluctuations in fluid membranes, whose only relevant degrees of freedom are the geometrical ones; I shall then briefly dwell on the effects of orientational order in hexatic\textsuperscript{2,3} and smectic\textsuperscript{4} membranes and I shall close by reporting the results of a recent work by David\textsuperscript{5} on the effect of the incompressibility constraint.

The geometry\textsuperscript{6} of a surface embedded in \(d\)-dimensional Euclidean space is defined by the mapping \(\tilde{\sigma} \rightarrow \mathbf{X}(\tilde{\sigma})\), where \(\tilde{\sigma} = (\sigma^1, \sigma^2)\) is the inner two-dimensional coordinate, and \(\mathbf{X} = (X^1, \ldots, X^d)\) is the corresponding location in bulk space. This geometry is locally described by two tensors: the metric tensor \(g_{ij}\) defines the square distance in bulk space between two points whose coordinates differ by \(ds\):

\[
ds^2 = g_{ij}d\sigma^id\sigma^j, \quad i, j = 1, 2. \tag{1}\]

The curvature tensor \(K_{ij}\) defines how the surface locally deviates from its tangent plane. It is most conveniently defined in terms of the covariant derivatives \(D_i\):

\[
K_{ij} = D_iD_j\mathbf{X} = \partial_i\partial_j\mathbf{X} - \Gamma^k_{ij}\partial_k\mathbf{X}, \tag{2}\]

where \(\Gamma^k_{ij}\) is a Christoffel symbol. Each component of this tensor is a vector in \(d\)-dimensional space, locally normal to the tangent plane. Therefore, for \(d = 3\) (and for an orientable surface), it may be represented just by a scalar. In the general case, one can build two Euclidean scalars out of it. The first is given by

\[
H^2 = K_i^i \cdot K_j^j. \tag{3}\]

In \(d = 3\) it is just the square of the sum of the inverse curvature radii. The second is given by

\[
2K = H^2 - K_i^i \cdot K_j^j. \tag{4}\]

This quantity is just the scalar curvature and may be expressed purely in terms of the metric tensor \(g_{ij}\).

The point to be kept in mind for fluid membranes is that all their physical properties should depend just on their geometry, and not on the particular mapping \(\tilde{\sigma} \rightarrow \mathbf{X}(\tilde{\sigma})\) we chose to represent it. By exploiting this argument, Helfrich\textsuperscript{1} showed...
that the free energy of an undulating membrane (neglecting membrane-membrane interactions) should be of the form

$$\mathcal{H} = \int d\sigma \sqrt{g} \left[ r_0 + \left( \frac{1}{2} \right) \kappa_0 H^2 + \bar{\kappa}_0 K + \text{higher order terms} \right].$$

We have defined $g = \text{Det}(g_{ij})$. The factor $d\sigma \sqrt{g}$ is just the area element on the membrane. In $d = 3$ there may appear, for membranes whose two sides are different, a term linear in $H$. The "higher order terms" involve higher derivatives of $X$. The coefficients $r_0$, $\kappa_0$, and $\bar{\kappa}_0$ are respectively called the (bare) surface tension, rigidity, and Gaussian rigidity. The third term does not play a role for small fluctuations of isolated membranes, since it may be shown that the integral $\int d\sigma \sqrt{g} \cdot K$ is a topological invariant. In usual situations, the first term is dominant. But it may be argued that the coefficient $r_0$ is rather small for fluctuating amphiphilic membranes.

For an almost planar membrane we may choose the coordinates $X$ and $\tilde{\sigma}$ such that $X = (\sigma^1, \sigma^2, h)$, where $h$ is a small, $(d - 2)$-dimensional vector. We can then expand Eq. (5) in powers of $h$, to obtain:

$$\mathcal{H} \approx \int d\tilde{\sigma} \left( \frac{1}{2} \right) \left[ r_0 (\partial_i h)^2 + \kappa_0 (\partial_i \partial_j h)^2 \right].$$

This tells us that the fluctuations of $\partial_i h$ are Gaussian, and are correlated up to distances of the order of $(\kappa_0/r_0)^{1/2}$. This distance diverges as $r_0$ goes to zero, what makes us expect that this limit should lead us to some sort of critical theory. Helfrich\textsuperscript{7} has drawn attention to this fact, what has encouraged a few investigators\textsuperscript{8-10} to look into the properties of such a model. Probably the clearest picture is obtained from the $d \to \infty$ calculation of Ref. 10, which I shall summarize here. We consider the membrane spanning a square frame of side $L$, and being immersed in a solution, thus fixing the chemical potential of the amphiphilic molecules, and therefore the coefficient $r_0$. Since the membrane area per molecule is more or less constant at equilibrium, the true area $S$ of the membrane is conveniently measured just by the number of molecules forming it. In general, we have $S \propto L^2$, with a proportionality constant $A$ which depends on $r_0, \kappa_0$, and the temperature $T$. We adopt units in which $kT = 1$. Then there appears to be a critical line $r_0 = r_c(\kappa_0)$ in the $(r_0, \kappa_0)$ plane at which the constant $A$ diverges. On this line $S$ increases faster than $L^2$, and one is tempted to associate with it a fractal dimension $d_f$ larger than 2. On the other hand, if one defines an effective rigidity by computing the energy needed to bend the frame, one finds that it decreases with increasing $L$—and actually (in the $d \to \infty$ limit) vanishes as $L$ diverges. Both these phenomena were anticipated by Helfrich\textsuperscript{7} and were confirmed by renormalized perturbation theory.\textsuperscript{8-9} One of the main consequences of the reduction of the effective rigidity is that the fluctuations in the tangents to the membrane are correlated—also along the critical line—only up to a finite persistence length $\xi$. In practice the rigidity does not have any effects on fluctuations of wavelength larger than $\xi$. As a consequence, the membrane is completely crumpled, and in the absence of self-avoidance, its fractal dimension equals infinity. This effect may also be seen as a restoration of Euclidean symmetry, in accordance with the Mermin-Wagner theorem: this symmetry is broken by the existence of the frame, but is restored very far away from it, near the center of the membrane, where no orientation of the tangent plane is preferred.