ESCAPE IN THE THREE-BODY PROBLEM WITH TWO DEGREES OF FREEDOM
-- RECTILINEAR CASE AND ISOSCELES CASE --

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ABSTRACT. With the Jacobi coordinates $r$, $R$, reduced masses $m$, $M$, and
the ratio $z = r/R$, the equations of motion and the energy of the system
are written by

$$\dot{r} = -\frac{B(z)}{r^2}, \quad \dot{R} = -\frac{A(z)}{R^2};$$

$$h = h_1 + h_2; \quad h_1 = m\left(\frac{1}{2} R^2 - \frac{B(z)}{r}\right), \quad h_2 = M\left(\frac{1}{2} R^2 - \frac{A(z)}{R}\right).$$

The functions $A(z)$ and $B(z)$ satisfy a relation $A(z)z^3 = B(z)$ according as
$z \leq k^*$, where $z = k^*$ corresponds to the central configurations. In the
region $z \leq k^*$ the sub-energies vary monotonically for every binary col­
lossion point ($r=0$) in succession (rectilinear case and isosceles case),
and for every point in succession with maximal value of $r$ (rectilinear
case), provided that $R > 0$ is satisfied throughout.

The motion is escape under the following initial conditions:

I. Rectilinear Case:
   i) $R > 0$ and $h_2 > 0$, at a point with $z \leq k^*$ and a maximal value of $r$;
   ii) $R > 0$, $h_2 > 0$, and $h_1 < 0$, at a point with $z = k^*$;
   iii) $R > 0$, $h_2 < 0$, and $h_1 > 0$, at a point with $z \geq k^*$;

II. Isosceles Case: $R > 0$, $h_2 > 0$, and $h_1 < 0$, at a binary collision point ($r=0$).

Finally, in the rectilinear case if $R > 0$ and $h_2 < 0$ at a binary col­
lossion point ($r=0$), then the trajectory in the $(R,r)$-space gets out of
the region $z \leq k^*$, otherwise the motion is ejection without escape.

1. PRELIMINARIES

1.1. Notations and Definitions

In the following the units are chosen to make the gravitational constant
equal to one. Let $m_1$, $m_2$, $m_3$ be the masses of three bodies and let $r$, $R$
be the Jacobi coordinates; $r$ is the distance of mass $m_2$ from mass $m_1$ and
$R$ is the distance of mass $m_3$ from the center of mass of $m_1$ and $m_2$. Now
we introduce the following notations:

\[ M \equiv m_1 + m_2 + m_3; \]
\[ m \equiv m_1 m_2 / (m_1 + m_2), \quad m_3 \equiv m_3 (m_1 + m_2) / M, \quad \text{(the reduced masses)}; \]
\[ \alpha \equiv m_1 / (m_1 + m_2), \quad \beta \equiv m_2 / (m_1 + m_2); \]
\[ z \equiv r / R, \quad \text{(the ratio of the distances)}. \]

In the rectilinear case, \( m_2 \) is assumed to be between \( m_1 \) and \( m_3 \), accordingly the ratio \( z \) satisfies \( 0 \leq z < \alpha^{-1} \). In the planar isosceles case we assume that \( m_1 = m_2 \) and \( r_{13} = r_{23} \), and we restrict our discussion to the case \( R \geq 0 \).

**Def. 1.1.** Let us define two functions \( A(z) \) and \( B(z) \) as follows:

1° In the rectilinear case,
\[
A(z) = M \left( \frac{\alpha}{(1-\alpha z)^2} + \frac{\beta}{(1+\beta z)^2} \right),
\]
\[
B(z) = (m_1 + m_2) - m_3 z^2 \left\{ \frac{1}{(1-\alpha z)^2} - \frac{1}{(1+\beta z)^2} \right\}.
\]

2° In the isosceles case,
\[
A(z) = 8 (2m_1 + m_3) \left( \frac{1}{(4+z^2)^{3/2}} \right),
\]
\[
B(z) = 2m_1 + \frac{8m_3 z^3}{(4+z^2)^{3/2}}.
\]

1.2. Equations of motion

In the general case, equations of motion are as follows:
\[
\frac{\dot{r}^2}{r} + \frac{m_1 + m_3}{r^3} \cdot \dot{r} = m_3 (\dot{r}_{13}^2 - \dot{r}_{13}^2), \quad \frac{\dot{r}}{R} = -M (\frac{\beta}{r_{13}^2} \cdot \dot{r}_{13} + \frac{\alpha}{r_{13}^2} \cdot \dot{r}_{13}).
\]

In particular, in our case with two degrees of freedom:
\[
\dot{r} = -\frac{B(z)}{r^2}, \quad \dot{r} = -\frac{A(z)}{R^2}.
\]

**Def. 1.2.** Let us define following quantities:

\[ h_1 \equiv m \left( \frac{1}{2} r^2 - \frac{B(z)}{r} \right), \quad h_2 \equiv M \left( \frac{1}{2} r^2 - \frac{A(z)}{R} \right). \]

These quantities can be uniquely defined at any binary collision point \((r=0)\), since the motion can be analytically continued. In addition, the energy integral of the system is denoted by
\[ h_1 + h_2 = h, \]
where \( h \) is a constant.

1.3. Fundamental properties of \( A(z) \) and \( B(z) \)

**Lemma 1.1.**

1° In the rectilinear case, for \( 0 \leq z < \alpha^{-1} \), the function \( A(z) \) is monotone increasing, and the function \( B(z) \) is monotone decreasing;

2° In the isosceles case, for \( z \geq 0 \), the function \( A(z) \) is monotone decreasing, and the function \( B(z) \) is monotone increasing.

**Lemma 1.2.** In both the cases we have following relations:

1° \( z^2 A(z) \geq B(z) \),
   for \( z = \kappa \ast \), where \( z = \kappa \ast \) corresponds to the central configurations.