CHAPTER ELEVEN

Transport by Advection

The basic theory of pollution transport by advection, dispersion, and diffusion, in groundwater, has been presented in Chapter 6. Numerical models for that general case are considered in Chapter 12. In this chapter, numerical models are presented for an approximate model (Section 6.6), in which the transport of a polluting component is taking place by advection only, neglecting transport by mechanical dispersion and molecular diffusion. In such a model the pollutant is transported at the average velocity of the water, which acts as its carrier.

Two numerical techniques are presented in this chapter: a semi-analytical method, based on an analytical solution of the groundwater flow problem, and a fully numerical method based on a finite element solution of the problem. In the latter case, it will turn out to be advantageous, from a viewpoint of accuracy, to use a formulation in terms of the stream function. All examples to be given in this chapter are basically two-dimensional, in the sense that one of the three components of the water velocity vector is so much smaller than either of the other two, that the velocity can be described by a vector in a plane.

11.1. Basic Equations

Before proceeding to the presentation of the numerical models, the basic equations for transport by advection only will be recalled from Chapter 6. As seen in that chapter, the basic transport phenomena of a tracer or a pollutant in groundwater consist of advection, which is transport at the average water velocity, and diffusion and dispersion, which constitute mechanisms for transport of the pollutant relative to the water. If the groundwater contains a single polluting component, the basic variable is its concentration in the water, which is denoted by $c$, the mass of tracer per unit volume of the pore water. The flux of this tracer, expressed as a mass flux through a unit area in the soil, is expressed by (6.2.19). Assuming that the soil is completely saturated with water, and neglecting dispersion and diffusion, this equation can be written as

$$q_c = ncV$$

(11.1.1)

where $n$ is the porosity of the soil, and $V$ is the average velocity of the groundwater.

The second basic equation is the equation of conservation of mass of the tracer (6.3.1). In the absence of losses due to adsorption, ion exchange, decay, etc., this
equation can be written as
\[ \frac{\partial (nc)}{\partial t} = -\nabla \cdot q_c. \] (11.1.2)

Substitution of (11.1.1) into (11.1.2) gives
\[ n \frac{\partial c}{\partial t} = -n \nabla \cdot \nabla c + c \nabla \cdot (n \nabla). \] (11.1.3)

Assuming that the flow is steady, the second term in the right-hand member vanishes and, thus, the basic equation becomes (see (6.6.8)),
\[ \frac{\partial c}{\partial t} = -V \cdot \nabla c = -V_x \frac{\partial c}{\partial x} - V_y \frac{\partial c}{\partial y} - V_z \frac{\partial c}{\partial z}. \] (11.1.4)

In the two-dimensional case, this equation further reduces to (6.6.9), i.e.,
\[ \frac{\partial c}{\partial t} = -V_x \frac{\partial c}{\partial x} - V_y \frac{\partial c}{\partial y}. \] (11.1.5)

This is the mathematical problem that is to be solved, with the concentration \( c \) being the unknown quantity, and the velocity field assumed to be given. It should be noted that this simplified problem has been obtained on the basis of a large amount of restrictive assumptions, such as steady flow, no dispersion, no adsorption, and no coupling through feedback of the tracer transport on the fluid transport. Thus, various important phenomena, including the influence of density differences, have been excluded. Nevertheless, it is believed that in many practical cases the major transport phenomenon is advection, so that at least the major component of the transport process is taken into account.

It was shown in Section 6.6 that an equation of the form (11.1.5) can be solved by the method of characteristics. The concentration remains constant in the direction of the characteristics, which coincides with the direction of the flow. Thus, in an infinitesimal time step \( dt \), the pollution is transported in spatial steps \( dx \) and \( dy \), such that
\[ dx = V_x \, dt, \quad dy = V_y \, dt. \] (11.1.6)

The main problem left to be solved now is to determine the streamlines. When these have been found, it still remains to trace the progress of the particles along them, in accordance with (11.1.6). For the determination of the streamlines, a semi-analytic method will be presented in the next section. A fully numerical solution will be presented in Section 11.6.

11.2. Semi-Analytic Solution

Analytic solutions have been obtained for a limited class of groundwater flow