CHAPTER NINE

The Finite Difference Method

In this chapter the finite difference method is presented, for problems of steady and nonsteady groundwater flow. The presentation will be oriented towards the introduction of simple computer programs, written in BASIC, that can be run on personal computers.

The finite difference method was the first method to be used for the systematic numerical solution of partial differential equations. Although the fundamental ideas have been established and used by mathematicians of the 18th century, such as Taylor and Lagrange, the application of the finite difference method to the solution of engineering problems is usually considered to be an achievement of 20th century scientists (Southwell, 1940; Kantorovich and Krylov, 1964).

In general, the method consists of an approximation of partial derivatives by algebraic expressions involving the values of the dependent variable at a limited number of selected points. As a result of the approximation, the partial differential equation describing the problem is replaced by a finite number of algebraic equations, written in terms of the values of the dependent variable at the selected points. The equations are linear if the original partial differential equations are also linear. The values at the selected points become the unknowns, rather than the continuous spatial distribution of the dependent variable. The system of algebraic equations must then be solved. This may involve a large number of arithmetic operations. Originally all these calculations were performed manually, or by using mechanical devices. However, since the advent of the electronic computer, they are usually executed by means of a computer program.

In recent years, other powerful numerical methods have been developed, in particular the finite element method. This method has a greater flexibility and, therefore, is often preferred to the finite difference method. Because the basic theory of the finite difference method is much simpler, and it usually requires less computer memory and computation time, this method will be presented first here. The finite element method will be presented in Chapter 10.

9.1. Steady Flow

To demonstrate the method, let us consider the case of two-dimensional flow of a single fluid in a homogeneous isotropic confined aquifer, with no sources or sinks. For this case, the flow is described by the Laplace equation (4.2.25b)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \quad (9.1.1)$$
This equation should be satisfied at all points within the considered aquifer domain \( R \). On the boundary of \( R \) the groundwater head, \( \phi \), should satisfy certain boundary conditions. Throughout this section it will be assumed that the boundary conditions are, following the discussion in Section 4.3

\[
\text{on } S_1: \quad \phi = f, \quad (9.1.2)
\]

\[
\text{on } S_2: \quad Q_n = -T \frac{\partial \phi}{\partial n} = 0 \quad (9.1.3)
\]

where \( S_1 \) and \( S_2 \) are disjoint parts of the boundary, which together form the entire boundary of the region \( R \). Equation (9.1.2) states that on the part \( S_1 \) of the boundary the head is prescribed. Equation (9.1.3) states that the remaining part of the boundary is impermeable. More general boundary conditions will be discussed later.

This basic idea of the finite difference method is that the region \( R \) is covered with two families of straight parallel lines, in the \( x \)- and \( y \)-directions, respectively, which together form a mesh of rectangles. The simplest type of mesh is generated when all the intervals between the lines are equal. In that case the mesh consists of squares (Figure 9.1). The value of the variable \( \phi \) in a nodal point of the mesh (or node) is now denoted by \( \phi_{i,j} \), where \( i \) indicates the position of the vertical mesh line (the column), and \( j \) the horizontal mesh line (the row).

The set of values \( \phi_{i,j} \) will be determined such that they approximate the continuous function \( \phi(x, y) \), at the points with coordinates \( x = x_i, \ y = y_j \). This can be accomplished by an approximation of the partial derivatives appearing in (9.1.1).

In general, the approximation of the first derivative with respect to \( x \) of a

![Fig. 9.1. A mesh of squares.](image-url)