MATRIX MECHANICS TO CLASSIFY NON-LINEAR CONTINUA

J.D. Coleman

Senior Visiting Fellow
Fluid Mechanics Division
Department of Civil Engineering
The City University, London.

SUMMARY

Matrix Mechanics is a procedure to classify linear, semi-linear and quasi-linear continua from the system of E first order partial differential equations in E dependent variables which define them. Purpose is to index problems, seek analogues and determine boundary conditions for higher order systems. The type assists choice of stable, efficient and convergent numerical procedures for the continuum. Dummy variables are used to reduce any second order terms to first order—one more variable—one more equation. System is arranged with each dependent variable in vertical rows. Each u say is replaced by C etc., where \( C(x, y, z, t) = \text{const.} \) is a characteristic for the system. The equation det \( A = 0 \) then defines all the E characteristics. The paper gives ground rules for determining characteristic classes. Then examples are systematically presented in eight different selected classes.

1. GROUND RULES FOR CLASSIFICATION

The equation det \( A = 0 \) usually splits up into a chain of homogeneous first and second order factors in the \( C_x, C_x, \) etc. Each first order factor gives one characteristic, each second order factor, two. If the original set was in \( x, y, z, t \), then a factor containing all the \( C_x, C_y, C_z, C_t \) is non-degenerate. If one or more are missing it is degenerate. Factors should be reduced to a sum of squares, but in some 30 practical examples so far, most have appeared already in that form. If any second order factor is (non-degenerate);
1) Positive or negative definite— it is elliptic, \([1]\).  
2) Indefinite with one odd sign— it is hyperbolic,  
3) Indefinite with two odd signs— it is ultrahyperbolic,  
4) If the factor is degenerate— it is parabolic.

G. N. Pande et al. (eds.), Numerical Techniques for Engineering Analysis and Design
© Martinus Nijhoff Publishers, Dordrecht 1987
Courant is quite clear on these points. So far, second order factors have appeared without multipliers to the \( C_x \) etc. and the sign problem is also unambiguous. Thus, a sum of squares, positive definite, in \( x, y, z \) is elliptic in \( x, y, z \) but is parabolic in \( x, y, z, t \). Rules for first order factors must be consistent with second order; it emerges that for consistency a non-degenerate first order factor is hyperbolic. If degenerate, it is parabolic. When all factors have been assigned to class, the class of the system is their sum in terms of \( E, H \) and \( P \). Numbers give the numbers of each type; thus 2H2P etc. We show multiplicity (overlap) of characteristics by brackets; thus 2H(2H)2P etc. Notably, characteristics can be found directly from the principal part of any eliminant of the system, but a theorem of Courant shows that that method may not give them all; the set may be incomplete. The set from the \( A \) matrix is complete. There is also an even and odd order theorem. Any totally elliptic, or elliptic (multiple) system has to be of even order; the roots are in pairs. Conversely, any odd order system has to have at least one real root; it is hyperbolic, or parabolic. Systems with only real roots, \( H \) or \( P \), can be of any order. Results so far show that elliptic and related systems are relatively few, hyperbolic and parabolic more common, but the most numerous engineering class is mixed hyperbolic/parabolic systems; mostly with multiple characteristics. A previous paper enumerated all possible non-degenerate classes for systems of order one to eight, \([2]\). Degenerate classes are legion. We now consider examples arranged systematically in eight selected classes; these examples are in the main drawn from the mechanics of the earth, ocean, air, sun and space. That is; environmental civil engineering. Probably infinitely complex, we approximate.

2. EXAMPLES

2.1 Equations which change type. Classification in a nonlinear system only has meaning at a given point, time, and for a given problem. Usually, systems do not change type over a usable field, but some do. These we now consider. To save space, we do not set out the bulky \( A \) matrix for each, but just give results. They can be checked using techniques given in the previous paper, \([2]\), and in Courant and Hilbert \([1]\). Two examples are:

\[ y \cdot u_{xx} + u_{yy} = 0 \text{ (Tricomi's equation)} \quad (1) \]

and

\[ (a^2 - (\phi_x)^2)\phi_{xx} - 2 \cdot \phi_x \cdot \phi_y \cdot \phi_{xy} + (a^2 - (\phi_y)^2)\phi_{yy} = 0 \quad (2) \]

both from gas dynamics-subsonic/transonic/supersonic gas flow, adiabatic and frictionless. In (1) the gas flow is vertical, downwards. Positive upper half plane, subsonic and elliptic. Lower half plane, supersonic and hyperbolic.