NORMAL FORMS WITH NOISE

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ABSTRACT. A method to eliminate the fast variables in stochastic differential equations near an instability is given. It is based in normal forms methods.

We shall review here some recent results concerning the elimination of fast variables in stochastic differential equations. This can be done working directly with the Fokker-Planck equation for the probability density [1, 2] or adapting to stochastic differential equations the methods used in deterministic equations. We shall proceed in the last way and prove a factorization theorem for the probability density of the stochastic process following [3].

We consider a general equation of the form

\[ \frac{d}{dt} \mathbf{U} = L \mathbf{U} + \mathbf{N} + \eta \xi(t) + \mathbf{L}^{(1)}(t) \mathbf{U} + \mathbf{M}(t, \mathbf{U}) \]  

(1)

where

\[ \mathbf{U} = \sum_{\alpha=1}^{N} U_{\alpha}(t) \mathbf{e}_{\alpha}, \quad \mathbf{e}_{1} = (1, 0, \ldots, 0), \ldots, \mathbf{e}_{N} = (0, \ldots, 0, 1), \]

and \( L \) is an \( N \times N \) matrix already in Jordan form. We suppose we have a critical space of dimension \( n \) generated by \( (\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}) \) and a stable space generated by \( (\mathbf{e}_{n+1}, \ldots, \mathbf{e}_{N}) \). One has

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\[
L_{\alpha} = \sum_{\beta=1}^{n} J_{\beta \alpha} e_{\beta}, \quad 1 \leq \alpha \leq n
\]

\[
L_{-\alpha} = \sigma e_{-\alpha}, \quad n + 1 \leq \alpha \leq N, \quad \sigma < 0,
\]
and \( J_{\alpha, \alpha+1} = 1 \) for some values of \( \alpha \). The diagonal terms are either zero or pure imaginary quantities corresponding to multiple Hopf bifurcations. We also have

\[
N(U) = \sum_{r \geq 2} N(r)(U)
\]

with

\[
N(r)(U) = \sum_{\alpha, \beta, \alpha = 1}^{N} u^{(r)}_{\alpha, \beta, \alpha_1 \ldots \alpha_r \beta_1 \ldots \beta_r} e_{\alpha}.
\]

The terms proportional to \( \eta \) in (1) are the noisy terms and the situation we have described up to now for the deterministic equations \( \partial_t U = L U + N(U) \) corresponds to be exactly at a critical point in the parameters space where one has a critical subspace of dimension \( n \) characterized by the Jordan matrix \( J_{\alpha \beta} \). The unfolding of this singularity can be obtained using the methods in [4] which are precisely the methods we shall generalize to the stochastic terms in (1). The latter are proportional to \( \eta \) which is a parameter measuring the intensity of the noise,

\[
D(t) = \sum_{\alpha=1}^{N} D(\alpha)e_{-\alpha},
\]

\( L^{(1)}(t) \) is an \( N \times N \) matrix

\[
L^{(1)}(t)e_{\alpha} = \sum_{\beta=1}^{N} L^{(1)}(t)_{\beta \alpha} e_{\beta} \quad \text{and} \quad M(t; U) = \sum_{r \geq 2} M^{(r)}(t; U)
\]

with

\[
M^{(r)}(t; U) = \sum_{\alpha, \beta, \alpha_1 \ldots \alpha_r} v^{(r)}_{\alpha, \beta, \alpha_1 \ldots \alpha_r} (t) U_{\alpha_1} \ldots U_{\alpha_r} e_{-\alpha}.
\]

Here \( \{D(\alpha), L^{(1)}(t)_{\alpha \beta}, v^{(r)}_{\alpha, \alpha_1 \ldots \alpha_r}(t)\} \) are taken as \( \delta \)-correlated gaussian