INVERSE SEMIGROUPS WITH COUNTABLE UNIVERSAL SEMILATTICES

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ABSTRACT. A semilattice $E$ is said to be a countable universal semilattice if $E$ is countable and if every countable semilattice can be embedded in $E$. The free Boolean algebra on a countably infinite number of generators is used to construct a particular countable universal semilattice which is the semilattice of idempotents of a 2-generated bisimple monoid.

Even though there are uncountably many pairwise non-isomorphic countable chains (consider the countable ordinals, for example), every countable chain can be embedded in the chain $(\mathbb{Q}, \leq)$ of rationals with the usual order. Therefore the chain $(\mathbb{Q}, \leq)$, being itself countable, is said to be a countable universal chain. On the other hand, there does not exist a countable universal group, that is, a countable group $G$ in which every countable group can be embedded. For such a group $G$ would have only a countable number of 2-generated subgroups, while by a result of B. H. Neumann [7] there exist uncountably many pairwise non-isomorphic 2-generated groups.

The chain $(\mathbb{Q}, \leq)$ has the additional property that any isomorphism between subchains of cardinality $<|\mathbb{Q}|$ can be extended to an automorphism of $(\mathbb{Q}, \leq)$; we say that $(\mathbb{Q}, \leq)$ is homogeneous. In fact, $(\mathbb{Q}, \leq)$ is the unique countable homogeneous universal chain. B. Jónsson [5,6] has formulated these notions in the context of relational structures, and has given sufficient conditions for a class of relational structures to possess a homogeneous universal structure of given cardinal. One of these conditions is the (weak) amalgamation property. We refer the reader to Chapter 10 of Bell and Slomson [1] for an exposition of Jónsson's theory.

Imaoka [4] has shown that the varieties of bands which have the (weak) amalgamation property are precisely those contained in the variety of normal bands. It therefore follows from Jónsson's results that any non-trivial variety of normal bands has a unique countable homogeneous universal element. In particular there exists a countable homogeneous universal semilattice. The countable homogeneous universal left [right] zero band $L [R]$ is just the countably infinite...
Our interest in countable universal semilattices is related to embedding theorems for inverse semigroups. N. R. Reilly [9] has shown that any inverse semigroup (and thus, in particular, any semilattice) can be embedded in some bisimple inverse semigroup. Chris Ash has shown that any countable inverse semigroup (and thus any countable semilattice) can be embedded in some 2-generated inverse semigroup. Ash's result appears in the survey article by T. E. Hall [3] in which these embedding theorems and many others are discussed in the context of amalgamation properties for inverse semigroups.

It is unknown whether the results of Reilly and Ash can be combined, that is, whether any countable inverse semigroup can be embedded in some 2-generated bisimple inverse semigroup. (It is known that any countable semigroup can be embedded in a 2-generated bisimple monoid [2]). We will show, however, that any countable semilattice can be embedded in some 2-generated bisimple inverse monoid. In fact, we prove a stronger result. We construct a countable universal semilattice which serves as the semilattice of idempotents of a 2-generated bisimple inverse monoid \( M \); thus any countable semilattice can be embedded in \( M \).

Our construction depends upon the duality between Boolean algebras and their Stone spaces, and on certain well-known topological properties of the Cantor set. We recall the relevant results (and refer the reader to [10] and [11] for example, for details) beginning with the Stone representation theorem for Boolean algebras: any Boolean algebra \( A \) is isomorphic to the Boolean algebra of all open-closed subsets of some totally disconnected compact Hausdorff space \( S(A) \). If \( A \) is countable, then the open-closed subsets of \( S(A) \) form a countable base and \( S(A) \) is metrizable. Thus any countable Boolean algebra \( A \) is isomorphic to the Boolean algebra of all open-closed subsets of some totally disconnected compact metric space. In particular, the free Boolean algebra \( \text{FB}_{\omega} \) on a countably infinite number of generators is isomorphic to the Boolean algebra of all open-closed subsets of the Cantor set \( \mathcal{C} \). But every compact metric space is a continuous image of \( \mathcal{C} \), and thus every countable Boolean algebra is embeddable in \( \text{FB}_{\omega} \).

The Cantor set \( \mathcal{C} \) is, up to homeomorphism, the only totally disconnected perfect compact metric space. In particular, any non-empty open-closed subset \( G \) of \( \mathcal{C} \) is homeomorphic to \( \mathcal{C} \). A homeomorphism from \( \mathcal{C} \) to \( G \) induces a Boolean isomorphism from \( \text{FB}_{\omega} \) onto the Boolean algebra of all open-closed subsets of \( G \) (i.e., onto the principal ideal of \( \text{FB}_{\omega} \) generated by \( G \)). Furthermore, if \( G \) and \( H \) are any open-closed subsets of \( \mathcal{C} \) which are proper and non-empty, then \( G \) and \( H \), as well as \( \mathcal{C} - G \) and \( \mathcal{C} - H \), are homeomorphic, and so there exists a homeomorphism of \( \mathcal{C} \) which maps \( G \) onto \( H \). This homeomorphism induces a Boolean automorphism of \( \text{FB}_{\omega} \) which takes \( G \) to \( H \).

The 2-generated bisimple inverse semigroup we seek will be obtained as a full inverse subsemigroup of the Munn semigroup \( T_E \) for an appropriate semilattice \( E \). (For standard results and