ON A NONLINEAR THEORY OF PHOTOELASTICITY

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As yet, in photoelasticity it has always been supposed Hooke's theory of elasticity to be valid and consequently linear relations between birefringence effects and stresses to be existing [1]. However, in areas of high stress concentration, e.g. in the vicinity of crack tips, notches and inclusions, and with respect to some of the mainly used photoelastic model materials, considerable uncertainties may result. Therefore the following nonlinear-elastic stress-strain relations will be introduced. On the supposition that the strains are still small, such relations may be formulated according to Kauderer [2] for an isothermal state:

$$\varepsilon_{ij} = \left[ \frac{1}{3K} \kappa(s) - \frac{1}{2G} g(\tau_o^2) \right] \delta_{ij} + \frac{1}{2G} g(\tau_o^2) \sigma_{ij},$$

(1)

where $\kappa(s)$ denotes a compression function, formulated as a potential series

$$\kappa(s) = 1 + \sum_{\nu=1}^{n} \frac{\kappa_{\nu}}{(3K)^{\nu}} s^{\nu},$$

(2)

and similarly, $g(\tau_o^2)$ denotes a shear function

$$g(\tau_o^2) = 1 + \sum_{\nu=1}^{n} \frac{g_{2\nu}}{(2G)^{2\nu}} \tau_o^{2\nu}.$$  

(3)

With $K$ and $G$, the initial values of the compression modulus and the shear modulus respectively, the coefficients $\kappa_{\nu}$ and $g_{2\nu}$ describe the nonlinear material response. These values are to be determined by material testing procedures, such as tensile tests and shear tests. Furthermore, in eqs (2) and (3), $s$ denotes the mean tension, and $\tau_o$ the reference stress. According to Neumann [3] (see also Coker/Filon [4], Mindlin [5]), in amorphous material the birefringence effect is assumed to be linear, depending on the mechanically induced strains. And of course there are some materials for which the validity of this assumption has been proved experimentally.

With $n$, the refraction index of the unstrained material, $n_{ij}$, the refraction tensor, and the strain-optical coefficients $d_1$ and $d_2$, the relation between birefringence and strain holds

$$n_{ij}^{-2} - n_{o}^{-2} \varepsilon_{ij} = d_1 \varepsilon_{ij} + \frac{1}{3} d_2 \varepsilon \delta_{ij}.$$

(4)

Wieringa, H (ed), Experimental Stress Analysis.
In eq. (4), \( e \) denotes the volume change and \( \varepsilon \) the strain deviation. Together with eq. (1), a nonlinear relation between the stress tensor and the refractive tensor will be obtained:

\[
\varepsilon_{ij} - \varepsilon_{0}^{-2} \delta_{ij} = A_{1} s \delta_{ij} + A_{2} \sigma_{ij} .
\]

(5)

For abbreviation it has been introduced

\[
A_{1} = d_{2} \frac{\kappa(s)}{3K} - d_{1} \frac{g(\omega_{s})}{2G} ; \quad A_{2} = d_{1} \frac{g(\omega_{s})}{2G} .
\]

In the following, a plane stress state may be considered in plane \((x_{1}, x_{2})\); the direction of the incident light ray may be parallel to the \( x_{3} \)-axis. Assuming a rectilinear light path, the component \( n_{33} \) of the refractive tensor is unessential to the further consideration, and because of \( \sigma_{13} = \sigma_{23} = 0 \), the components \( n_{13} \) and \( n_{23} \) are also equal zero. Thus, the refractive tensor is reduced to \( n_{\alpha \beta} \), \( \alpha, \beta \in \{1, 2\} \). The principal axes \( \psi \) of the index ellipse with reference to the \( x_{1} \)-axis are given by the proper vectors of \( n_{\alpha \beta}^{-2} \):

\[
\tan 2\psi = 2 n_{12}^{-2} \left[ n_{11}^{-2} - n_{22}^{-2} \right]^{-1} .
\]

(6)

It can be proved easily, that the principal directions of the index ellipse coincide with the principal axes of the stress tensor. The eigenvalues of \( n_{\alpha \beta}^{-2} \) are the principal values \( n_{\alpha}^{-2} \); they are given by

\[
\begin{align*}
\n_{1}^{-2} &= n_{0}^{-2} + \left( A_{1} + \frac{3}{2} A_{2} \right) s + \frac{1}{2} A_{2} \left[ (\sigma_{11} - \sigma_{22}) \cos 2\psi_{N} + 2 \sigma_{12} \sin 2\psi_{N} \right] , \\
\n_{2}^{-2} &= n_{0}^{-2} + \left( A_{1} + \frac{3}{2} A_{2} \right) s - \frac{1}{2} A_{2} \left[ (\sigma_{11} - \sigma_{22}) \cos 2\psi_{N} + 2 \sigma_{12} \sin 2\psi_{N} \right] .
\end{align*}
\]

(7)

Because of the weak birefringence, from eq.s (7) follows

\[
\begin{align*}
\n_{1} - \n_{2} &= - \frac{1}{2} n_{0}^{3} A_{2} \left[ (\sigma_{11} - \sigma_{22}) \cos 2\psi_{N} + 2 \sigma_{12} \sin 2\psi_{N} \right] , \\
\end{align*}
\]

(8)

where \( n_{\alpha} \) denote the refraction indices.

The birefringence is defined as the difference of the refraction indices \( n_{\alpha} \)

\[
\Delta := n_{1} - n_{2} ;
\]

(9)

and by the relation

\[
\Delta = \frac{\lambda}{d} \delta ,
\]

(10)

related to the wave length of the used monochromatic light, the thickness \( d \)