13. Some special games and methods

In the previous chapter we dealt with methods capable of solving arbitrary matrix games. In practice, however, we often come across games having some special features enabling us to use simpler methods to solve them and/or their size can be considerably reduced. Thereby they are more amenable to numerical treatment. In this chapter we are going to discuss some special games and methods of this sort.

13.1 MATRICES WITH SADDLE-POINTS

The simplest matrix games are those having an equilibrium point in pure strategies. In this case there are indices $i_0, j_0$ such that $x^0 = e_{i_0}$ and $y^0 = e_{j_0}$ are equilibrium strategies. By Theorem 3 of Chapter 9 we have

\[ x^0 A = e_{i_0} A \geq v1 = a_{i_0, j_0} 1, \]
\[ A y^0 = A e_{j_0} \leq v1 = a_{i_0, j_0} 1. \]

This means that $a_{i_0, j_0}$ is minimal in the $i_0$th row and maximal in the $j_0$th column. We call $a_{i_0, j_0} = v$ the saddle point of the matrix game $A$.

Conceptionally, pure strategies provide the most acceptable solutions particularly in situations where the game cannot be played several times. When analyzing practical game situations the first thing we have to do is check for saddle-points. This can very easily be done by inspection.

Unfortunately if we pick "randomly" from all the possible $m \times n$ game matrices, then it is very improbable that we will get a matrix...
with a saddle-point. The following theorem makes this statement precise.

**THEOREM 1. [61]** Let \( A \) be an \( m \) by \( n \) matrix each element of which is a random variable of the same continuous distribution. Then the probability that \( A \) has a saddle-point is

\[
P(m, n) = \frac{m! \ n!}{(m + n - 1)!}.
\]

**Proof.** Consider the events

- \( E \): \( A \) has a saddle-point,
- \( E(r, s): a_{rs} \) is a saddle-point of \( A \).
- \( F \): \( A \) has all its elements distinct.

Our continuity assumption ensures \( P(F) = 1 \), so that

\[
P(m, n) = P(E) = P(E \cap F).
\]

Since the value of a game is unique if \( a_{rs} \) and \( a_{tu} \) are both saddle-points, then \( a_{rs} = a_{tu} \). Thus \( E \cap F \) is the disjoint union of the events \( E(r, s) \cap F \), and so

\[
P(m, n) = \sum_{r=1}^{m} \sum_{s=1}^{n} P[E(r, s) \cap F].
\]

By symmetry \( P[E(r, s) \cap F] \) has the same value for all pairs \((r, s)\) and thus

\[
P(m, n) = mnP[E(1, 1) \cap F].
\]

The \((m + n - 1)!\) possible orderings of the elements of the first row and column are clearly equiprobable and exactly \((m - 1)!(n - 1)!\) of them make \( a_{11} \) a saddle-point. Therefore

\[
P[E(1, 1) \cap F] = \frac{(m - 1)!(n - 1)!}{(m + n - 1)!},
\]

implying the validity of our theorem.