3. Existence theorems of equilibrium points

Let a game $\Gamma$ be given in normal form

$$\Gamma = \{\Sigma_1, \ldots, \Sigma_n; K_1, \ldots, K_n; \Phi_1, \ldots, \Phi_n\}.$$ 

As we have already seen an equilibrium point of the game $\Gamma$ is an $n$-tuple of strategies $(\sigma_1^*, \ldots, \sigma_n^*)$, $(\sigma_i^* \in \Sigma_i, i = 1, \ldots, n)$ for which

(a) $\sigma_i^* \in \Phi_i(\sigma_1^*, \ldots, \sigma_n^*)$, $(i = 1, \ldots, n)$,
(b) $K_i(\sigma_1^*, \ldots, \sigma_{i-1}^*, \sigma_i, \sigma_{i+1}^*, \ldots, \sigma_n^*) \leq K_i(\sigma_1^*, \ldots, \sigma_n^*)$

for each $\sigma_i \in \Phi_i(\sigma_1^*, \ldots, \sigma_n^*)$, $(i = 1, \ldots, n)$.

The equilibrium point thus defined is generally referred to as a (generalized) Nash-equilibrium point.

Very often we have to deal with games where the condensed information supplied by the normal form is not enough to describe the behaviour of the players, the rules of the game etc. in detail. As a first step to formalizing an actual game the so-called extensive form seems to be more adequate. An $n$-person game $\Gamma$ is said to be given in extensive form if a directed rooted tree (finite or infinite) can be associated with it\(^1\) by the following properties:

(a) The game starts at the root of the tree.
(b) To any node of the tree a player is assigned and the game proceeds to the vertex chosen by that particular player.
(c) Each player knows the vertices at which he has to make a decision.

\(^1\) In the following — unless otherwise stated — we shall deal only with extensive games given by finite trees.
(d) There are given functions $f_1, \ldots, f_n$ assigning pay-offs to any player at each terminal node of the tree.

The game is said to be of complete information if each player when making his move knows exactly at which node he is and remembers perfectly how the game got to that particular node.

An additional "player", a "random mechanism" called "chance" may also participate in the game. If it is chance's turn to move, then the game gets to the next node according to a probability distribution. Chance is not a conscious player thus its behaviour is not governed by a strategy but by a probability distribution. For player $i$ (except for "chance"), a strategy $\sigma_i$ is a function defined on the nodes where he is to move and tells him how to proceed if the game ever gets to the particular node. Thus if chance is not involved in the game, then an $n$-tuple of strategies $(\sigma_1, \ldots, \sigma_n)$ uniquely determines a terminal node of the tree where each player gets $f_i(\sigma_1, \ldots, \sigma_n)$ as pay-offs. If chance is one of the players, then each terminal node $V_j$ ($j = 1, \ldots, N$) is reached with some probability $p_j(\sigma_1, \ldots, \sigma_n)$. Obviously

$$\sum_{j=1}^{N} p_j(\sigma_1, \ldots, \sigma_n) = 1.$$  

In this case the expected pay-offs

$$K_i(\sigma_1, \ldots, \sigma_n) = \sum_{j=1}^{N} f_i^{(j)} p_j(\sigma_1, \ldots, \sigma_n), \quad (i = 1, \ldots, n)$$

are considered as pay-off functions, where $f_i^{(j)}$ denotes the pay-off of player $i$ at the $j$th terminal node.

Let us now form subsets $H_1, \ldots, H_s$ of players in such a way that each player belongs to at least one $H_i$. Denoting by $n_i$ the cardinality of $H_i$, $(i = 1, \ldots, s)$ we have $\sum_{i=1}^{n} n_i \geq n$. If $H_i \cap H_j = \emptyset$, $(i \neq j)$, then we speak about a partition. An $n$-tuple of strategies $(\sigma_1^*, \ldots, \sigma_n^*)$ is said to be a group equilibrium point if