SURFACE SPLINE INTERPOLATION: BASIC THEORY AND COMPUTATIONAL ASPECTS

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ABSTRACT

One of the most efficient methods to date for global interpolation of scattered data has come to be called "surface spline interpolation". It turns out that the underlying mathematical theory has for natural setting some functional semi-Hilbert space whose reproducing kernel is known in closed form and can be computed economically. Solving the interpolation problem thus amounts to minimizing some Sobolev seminorm under interpolatory constraints or eventually, to solving a positive definite linear system. Our purpose here is to give a motivated and self-contained presentation of this interesting material.

1. INTRODUCTION

The problem of fitting appropriate surfaces to arbitrarily spaced data arises in many applications in science and technology. A variety of numerical methods have been devised for representing and constructing such surfaces. For a survey, see e.g. the "classical" references [1, 13] and the comprehensive technical report [4] (condensed in the recent paper [5]), which is essentially devoted to a critical comparison (in terms of timing, storage, accuracy, visual pleasantness, ease of implementation, etc.) of some 29 methods (either local or global in nature) for interpolation of scattered data.

It turns out that the particular method of surface spline interpolation (also known as thin plate spline interpolation)
we will here discuss thoroughly is not only one of the most efficient methods to date for global interpolation (see [4], p. 82), but also one of the few methods to derive from a truly elegant mathematical structure which is perfectly understood by now (see e.g. [3] and, for a deliberately more constructive approach, [7, 8, 9] and also [10, 11]). This paper is devoted to a motivated, self-contained presentation of that most interesting method. As regards the underlying mathematical theory (see Section 2), it can be said that a prominent role is played by the deep Theorem 1, which is given here (for the first time) a reasonably simple, constructive proof. Owing to this basic theorem, the interpolants to be considered must belong to a functional semi-Hilbert space of continuous functions, the reproducing kernel of which is explicitly known and involves no functions more complicated than logarithms (see Section 3). It follows that solving the interpolation problem amounts strictly to solving a Cramer system of linear equations with a Gram matrix (whose elements, just like the reproducing kernel itself, can be easily computed).

The solution of a linear algebraic system with a positive definite symmetric matrix of coefficients (which is precisely the case here) is a quite simple problem (at least in principle). Indeed, the classical Cholesky algorithm is very economical (in comparison with Gaussian elimination, with suitable pivotal strategy, for more general matrices) and, moreover, enjoys remarkable numerical stability. Though this most important fact has been known to numerical analysts for a long time, a detailed rounding-error analysis of the Cholesky process in standard floating-point arithmetic is essentially lacking in the current technical literature, the very recent paper [12] excepted. In Section 4, we will summarize (without proof) the main results of the two self-contained round-off analyses presented in [12], to wit: novel a posteriori error bounds (of practical use) and refined a priori results (of more theoretical significance, see also [14]).

For simplicity, all functions and vector spaces considered in this paper are real.

2. THE BASIC INTERPOLATION PROBLEM

Roughly speaking, the interpolation problem to be addressed here can be formulated as follows: Given a finite set \( A = \{a_i\}_{i \in I} \) of distinct points of \( \mathbb{R}^n \) and associate (real) values \( \{a_i\}_{i \in I} \), construct a (continuous) function \( v: \mathbb{R}^n \to \mathbb{R} \) so that \( v(a_i) = a_i \), \( i \in I \).