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THE INFLUENCE OF HERACLITUS ON MODERN MATHEMATICS

Among the pre-Socratic philosophers we know that Thales, Pythagoras, Zeno and Democritus were involved in mathematics in one way or another. In fact, Pythagoras coined the very word "mathematics." Heraclitus does not appear to have occupied himself with mathematical questions; so how can he be said to have influenced modern mathematics? The answer to this question will take us on a small detour.

What has survived of Heraclitus' thought is contained in a number of pithy statements. I shall only mention two:

> "You cannot step into the same river twice",

and

> "Men do not know how what is at variance agrees with itself; it is an attunement of opposite tensions, as in the bow or in the lyre".

Of all the Greek philosophers, Heraclitus has perhaps been the most influential in the twentieth century. When Mao Tse-tung refers to the "law of contradiction in things" as "the unity of opposites," he is quoting indirectly from Heraclitus. How did this come about?

The nineteenth century German philosopher Hegel developed the ideas of Heraclitus into his theory of dialectics, viewing the universe as a sort of divine debating society in which opposing ideas are forever struggling to produce a synthesis. Marx and Engels took up Hegel's dialectics and gave it a materialistic basis. Marx himself had studied Democritus for his doctoral dissertation, comparing him with Epicurus, and mentioned Heraclitus only marginally. However, both Engels and Lenin acknowledged that dialectic materialism originated in the thought of Heraclitus, as has been eloquently argued and amply documanted by Kessidi.

At first it seems reasonable to expect to find the dialectical process in history, economics, biology and perhaps other sciences. But mathematics is supposed to deal with eternal and unchanging objects such as the number 5. Indeed, when later dialecticians rephrased the Heraclitian doctrine of continual change by saying that "*a* is not always equal to *a*,"

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this appears to contradict a fundamental axiom of mathematics and is, in fact, not taken seriously by practicing mathematicians in communist countries.

However, Heraclitus' doctrine of the unity of opposites has occasionally been illustrated by examples from mathematics. Lenin asserted somewhere – and I quote from memory – that subtraction is the antithesis of addition, with arithmetic as the synthesis, and that integration is the antithesis of differentiation, with calculus as the synthesis. I did find a relevant quotation by Mao Tse-tung:

> The classification of scientific studies is based precisely upon the particular contradictions inherent in their objects .... For example, positive numbers and negative numbers in mathematics, action and reaction is mechanics,....

Other mathematical illustrations were not so fortunate. I recall one Marxist illustrating the change of quantity into quality by pointing out that the number 2 is even, instead of being twice as odd as the number 1.

I spent my sabbatical 1965–6 in Zürich, where I had many conversations with the young American mathematician, Bill Lawvere. We kicked around the idea that an interesting illustration of dialectic contradictions could be found in the "adjoint functors" of modern mathematics, which had recently been popularized by Peter Freyd in his book on Abelian categories. While the basic idea behind adjoint functors is fully contained in the "universal mapping problem" of Nicolas Bourbaki, their more symmetric presentation by Kan in 1958 relied on the notion of a category invented by Eilenberg and MacLane.

To me the connection between adjoint functors and dialectical contradictions was not much more than an amusing analogy, but to Bill Lawvere it was of profound significance and pervaded the whole structure and development of mathematics. Having been influenced by General Semantics in his youth, he was now becoming attracted to Marxism-Leninism and came to see the dialectical process everywhere in mathematics. When asked point blank years later he would not admit that adjoint functors were the only manifestations of contradictions in mathematics. However, in this essay, I shall confine myself mainly to this narrower view.

In his 1970 address on 'Quantifiers and Sheaves' to the International Congress of Mathematicians in Nice, Lawvere said the following:

> The unity of opposites... is essentially that between logic and geometry, ... We first sum up the principal contradictions of the Grothendieck-Giraud-Verdier theory of