Using Oseledec' multiplicative ergodic theorem we prove the existence of a fundamental system of solutions of \( \dot{x} = A(t)x, \) \( A(\cdot) \) stationary, allowing a Floquet type decomposition into a stationary angular part and a growing radial one. For triangular matrices the decomposition is explicitly calculated as a functional of \( A. \)

1. THE PROBLEM

We consider the linear vector differential equation

\[
\dot{x} = A(t)x
\]

where \( A(\cdot) \) is "real noise", i.e. an \( n \times n \)-matrix valued stochastically right continuous strict sense stationary (and ergodic) process with \( E|A(0)| < \infty. \) How does the solution \( x_t = \Phi(t)x_0, \Phi(t) \) fundamental matrix, reflect the stationarity of the noise? In the deterministically degenerated case with constant \( A(t,\omega) = A \) the fundamental matrix \( \Phi(t) = \text{Id} \exp(tA) \) is decomposed into a constant, thus stationary matrix \( \text{Id} \) and a part describing the growth; more generally, for deterministic periodic matrices \( A(t) = A(t+T) \) by Floquet theory we have \( \Phi(t) = P(t)\exp(tR) \) with differentiable periodic \( P(t) = P(t+T) \) and constant \( R. \)
Our stochastic problem is to find a fundamental system $\Phi(t)$ with an analogous decomposition such that

$$\Phi(t) = S(t)\exp(\Lambda t + o(t))$$

where $S(\cdot)$ is a differentiable stationary matrix valued process and $\Lambda$ is constant with a spectrum consisting of all possible orders of growth defined by the Lyapunov numbers

$$\lambda(x) = \lim_{t \to \infty} \frac{1}{t} \log |\Phi(t)x|, \Phi(t) \text{ fundamental matrix with } \Phi(o) = Id$$

2. ORDER OF GROWTH

Let $(\Omega, \mathcal{F}, P)$ be the canonical probability space of the stationary noise $A(\cdot)$, $(\Theta_t)_{t \in \mathbb{R}}$ the associated group of measure preserving shifts. Then the fundamental matrix $\Phi(t)$ is an $A$-cocycle, i.e. it is $A$-measurable and

$$\Phi(t+s, \omega) = \Phi(t, \Theta_s \omega) \Phi(s, \omega).$$

Moreover, since $A(t) \in L^1(\Omega, P)$, we have (see OSELEDEC [5])

$$\sup_{-1 \leq s \leq 1} \log \|\Phi(s)\| \in L^1(\Omega, P).$$

**Theorem 1** (Multiplicative ergodic theorem of Oseledec)

Under the above conditions we have $P$-almost surely:

$$\mathbb{R}^n = \bigoplus_{i=1}^{r(\omega)} E_i(\omega), \quad k_i(\omega) = \dim E_i(\omega),$$

with the properties:

(i) there are $r(\omega)$ Lyapunov numbers $\lambda_1(\omega) < \ldots < \lambda_r(\omega)$ such that

$$\lambda_i(\omega) = \lambda_i(\omega, x) = \lim_{t \to \infty} \frac{1}{t} \log |\Phi(t, \omega)x|$$

uniformly for all $x \in E_i(\omega)$, $i = 1, \ldots, r(\omega)$;

(ii) the random variables $r$, $k_i$, $\lambda_i$ are invariant with respect to $(\Theta_t)$ and $E_i(\Theta_t\omega) = \Phi(t)(\omega)E_i(\omega)$;

thus if $A$ is stationary and ergodic $r$, $k_i$ and $\lambda_i$ are constants.

For proofs see OSELEDEC [5], RAGHUNATHAN [6] and RUELLE [7].

In the following we always assume ergodicity of $A$. So if $x_0(\omega) \in E_i(\omega)$, by projection onto the unit sphere $S^{n-1}$, we have $x_t = w_t|x_t| = w_t \exp(t\lambda_i + o(t))$, where