A TUTORIAL INTRODUCTION TO DIFFERENTIABLE MANIFOLDS AND VECTOR FIELDS

Michiel Hazewinkel
Dept. Math., Erasmus Univ. Rotterdam

In this tutorial I try by means of several examples to illustrate the basic definitions and concepts of differentiable manifolds. There are few proofs (not that there are ever many at this level of the theory). This material should be sufficient to understand the use made of these concepts in the other contributions in this volume, or, at least, it should help in explaining the terminology employed.

1. INTRODUCTION AND A FEW MOTIVATIONAL REMARKS

Roughly an n-dimensional differentiable manifold is a gadget which locally looks like $\mathbb{R}^n$ but globally perhaps not; A precise definition is given below in section 2. Examples are the sphere and the torus, which are both locally like $\mathbb{R}^2$ but differ globally from $\mathbb{R}^2$ and from each other.

Such objects often arise naturally when discussing problems in analysis (e.g. differential equations) and elsewhere in mathematics and its applications. A few advantages which may come about by doing analysis on manifolds rather than just on $\mathbb{R}^n$ are briefly discussed below.
1.1 Coordinate freeness ("Diffeomorphisms"). A differentiable manifold can be viewed as consisting of pieces of $\mathbb{R}^n$ which are glued together in a smooth (= differentiable) manner. And it is on the basis of such a picture that the analysis (e.g. the study of differential equations) often proceeds. This brings more than a mere extension of analysis on $\mathbb{R}^n$ to analysis on spheres, tori, projective spaces and the like; it stresses the "coordinate free approach", i.e. the formulation of problems and concepts in terms which are invariant under (nonlinear) smooth coordinate transformations and thus also helps to bring about a better understanding even of analysis on $\mathbb{R}^n$. The more important results, concepts and definitions tend to be "coordinate free".

1.2 Analytic continuation. A convergent power series in one complex variable is a rather simple object. It is considerably more difficult to obtain an understanding of the collection of all analytic continuations of a given power series, especially because analytic continuation along a full circle may yield a different function value than the initial one. The fact that the various continuations fit together to form a Riemann surface (a certain kind of 2-dimensional manifold usually different from $\mathbb{R}^2$) was a major and most enlightening discovery which contributes a great deal to our understanding.

1.3 Submanifolds. Consider an equation $\dot{x} = f(x)$ in $\mathbb{R}^n$. Then it often happens, especially in problems coming from mechanics, that the equation is such that it evolves in such a way that certain quantities (e.g. energy, angular momentum) are conserved. Thus the equation really evolves on a subset $\{x \in \mathbb{R}^n | E(x) = c\}$ which is often a differentiable submanifold. Thus it could happen that $\dot{x} = f(x)$, $f$ smooth, is constrained to move on a 2-sphere which then immediately tells us that there is an equilibrium point.