Chapter 2

THE LAWSON TOPOLOGY ON QUASICONTINUOUS DOMAINS *

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Abstract  For a directed complete poset $P$, let $\lambda(P)$ and $\sigma(P)$ be the lower topology and the Lawson topology on $P$ respectively. We constructively prove that if $P$ is a quasicontinuous domain and all lower closed subsets in $(P, \lambda(P))$ are closed in $(P, \omega(P))$, then $(P, \lambda(P))$ is strictly completely regular ordered space.

Keywords: Quasicontinuous domain, Meet-continuity, Lawson topology, Strictly complete regularity, Hausdorff separation

1. Introduction

In the recent twenty years, the generalizations of continuous lattices (posets) have attracted a considerable deal of attention. The generalized continuous lattices and the quasicontinuous domains (see [GL81, GLS83, VegO]) are among the most successful such generalizations. For continuous domains, Lawson [La91] proposed the following:

Problem. Let $P$ be a continuous domain endowed with the Lawson topology. Is $P$ a strictly completely regular ordered space?

We prove constructively, in the theory ZFDC$\omega$, that if $P$ is a quasicontinuous domain and all lower closed subsets in $(P, \lambda(P))$ are closed

*Project supported by the National Natural Science Foundation of China and the Natural Science Foundation of Jiangxi Province, China

in \((P, \omega(P))\), then \((P, \lambda(P))\) is strictly completely regular ordered space. Also in the paper, we show that the following three conditions are equivalent: (1) \(P\) is a continuous domain; (2) \(P\) is a meet-continuous and quasicontinuous domain; and (3) \(P\) is a meet-continuous domain in which \(\downarrow x\) is directed for each \(x \in P\) and the Lawson topology \(\lambda(P)\) is Hausdorff.

2. Quasicontinuous domains

The terminology and notation used here are standard and well known. The reader wishing more details can consult [COMP], [GL81], and [GLS83]. In what follows the symbol \(\omega\) denotes the ordinal number of the set of natural numbers.

Let \(P\) be a poset. For \(x \in P\) and \(A \subseteq P\), we define \(\downarrow x = \{y \in P : y \leq x\}\) and \(\downarrow A = \bigcup\{\downarrow a : a \in A\}\); dually, we define \(\uparrow x\) and \(\uparrow A\). Let \(P^{(\omega)} = \{F \subseteq P : F \text{ is finite}\}\).

The topology generated by the collection of sets \(P \setminus \uparrow x\) (as subbasic open subsets) is called the lower topology on \(P\) and denoted by \(\omega(P)\). If \(P\) is a dcpo (that is, a directed complete poset), then we can define a topology \(\sigma(P)\) by \(U \in \sigma(P) \iff U = \uparrow U\) and for each directed set \(D \subseteq P, \forall D \in U \Rightarrow D \cap U \neq \emptyset\). The topology \(\sigma(L)\) is called the Scott topology on \(P\). The topology \(\lambda(P) = \omega(P) \lor \sigma(P)\) is called the Lawson topology on \(P\). Let \(\lambda(P)^{\uparrow} = \{U \in \lambda(P) : U = \uparrow U\}\) and \(\sigma(P)^{\uparrow} = \{U \in \lambda(P) : U = \downarrow U\}\). Clearly, \(\omega(P) \subseteq \lambda(P)^{\uparrow}\) and \(\sigma(P) \subseteq \lambda(P)^{\uparrow}\).

**Definition 2.1.** Let \(P\) be a dcpo and \(A, B \subseteq P\).

(1) We say \(A\) is way below \(B\), written \(A \ll B\), if for each directed subset \(D \subseteq P, \forall D \in \uparrow B \Rightarrow D \cap \uparrow A \neq \emptyset\). For \(B = \{x\},\) a singleton, \(A \ll B\) is written \(A \ll x\) for short. For \(x \in L\), let \(w(x) = \{F \in P^{(\omega)} : F \ll \{x\}\}\) and let \(\downarrow x = \{y \in P : y \ll x\}\).

(2) \(P\) is called a continuous domain, if for each \(x \in P, \downarrow x\) is directed and \(x = \lor \downarrow x\).

(3) \(P\) is called a quasicontinuous domain, if for each \(x \in P, \uparrow x = \bigcap\{\uparrow F : F \in w(x)\}\) and the family \(\{\uparrow F : F \in w(x)\}\) is filtered. In case of complete lattices, the notions of quasicontinuity and generalized continuity (see [GL81]) agree.

By considering singleton directed subsets of \(P\), we have that \(A \ll B \Rightarrow \uparrow B \subseteq \uparrow A\). For \(A \subseteq P\), let \(\uparrow A = \{x \in P : A \ll x\}\).

**Lemma 2.1.** (Gierz-Lawson-Stralka [GLS83]) Let \(P\) be a quasicontinuous domain. Then

1. The way below relation \(\ll\) satisfies the following interpolation:

\[(\text{INT}) \forall F, H \in P^{(\omega)}[F \ll H \Rightarrow \exists G \in P^{(\omega)} \text{ with } F \ll G \ll H].\]