In this chapter, we study some basic entailment type systems. Let $A_1, \ldots, A_n \rightarrow B_1, \ldots, B_m$ denote an $n+m$ place connective and let $(S, R, o, D_t)$ be an EN structure (i.e. with $R \subseteq S^{n+m+1}$) (see (35.14)). We say that $\rightarrow$ has an entailment type truth table in the structure (EE truth table) if we have

$$\| A_1, \ldots, A_n \rightarrow B_1, \ldots, B_m \|_{EE} = 1$$

iff

for all $s_1, \ldots, s_n, t_1, \ldots, t_m$ if

$$(t, s_1, \ldots, s_n, t_1, \ldots, t_m) \in R$$

and for all $1 \leq i \leq n$

$$\| A_i \|_{EE} = 1$$

then

$$\| B_j \|_{EE} = 1, \text{ for some } 1 \leq j \leq m.$$  

For languages containing classical negation, $\| \|_{EE}$ coincides with $\| \|_E$ of (35.21) compare also with (35.28) and (10.12).

In this chapter, we study basic systems containing EE type operators of the form $A \rightarrow B$. The language does not contain classical implication, $\rightarrow$, at all (originally entailment logic were supposed to replace $\rightarrow$ by $\rightarrow$, avoiding various undesirable properties of material implication). Classical negation is also missing from these systems. To handle the systems of this chapter, we must resort to the notion of the consequence relation of Chapter 0, and related concepts. In the rest of this introduction we recall briefly the necessary concepts from Chapter 0, for languages containing the classical $\land$ and $\lor$.

Given a language and a semantical interpretation for the language (we shall be more precise later, meanwhile take it to mean a class of EE type structures), we would like to write a logical system complete for this interpretation. What one usually does in the literature is to give axioms and rules of inference, and then show that the completeness theorem holds. For many languages however, particularly those without material implication this is not sufficient. For example there may be (see below) two distinct interpretations that have no theorems (i.e. every sentence $A$, in both interpretations, has both, a structure where it is valid and a structure where it is not valid) and therefore cannot be characterized by a mere set of axioms and rules. This leads one to turn to the general notion of the consequence relation and use this notion to characterize various interesting semantical interpretations.

The basic concepts involved are the following: An intensional propositional language contains, besides a set $P$ of propositional variables, a set $C$ of sentential connectives, or operators, which includes two kinds of connectives:

(a) Some, possibly none, of the following classical connectives: $t, f, \neg, \land, \lor$, and $\rightarrow$.

(b) Several additional $n$-ry intensional connectives, such as $\neg$, $I$, (unary); $\rightarrow$ (binary); $\epsilon_m$ ($m$-ry) etc.

In the sequel we assume that we have a fixed intensional language, with a set $W$ of wff. We now proceed to the basic notions of a logical system and semantics studied below.
A semantics for this language is a set \( T \subseteq 2^W \) with the property that each \( s \in T \) is compatible with the classical connectives of \( C \), i.e. the following holds:

(a) \( s(t) = 1; \quad s(f) = 0 \).
(b) \( s(A \land B) = 1 \) if \( s(A) = 1 \) and \( s(B) = 1 \).
(c) \( s(A \lor B) = 1 \) if \( s(A) = 1 \) or \( s(B) = 1 \).
(d) \( s(\neg A) = 1 \) if \( s(A) = 0 \).
(e) \( s(A \rightarrow B) = 1 \) if \( s(A) = 0 \) or \( s(B) = 1 \).

An \( s \in 2^W \) with the above mentioned properties is called a model and so a semantics \( T \) is a class of models. This is the most general semantical interpretation for the connectives of \( C \).

Given a semantics \( T \), we define the following (consequence) relation \( \vdash_T \) associated with \( T \) by:

\[ A \vdash_T B \text{ iff for all } s \in T, \text{ if } s(A) = 1 \text{ then } s(B) = 1. \]

Generally, for languages with classical \( \land \) and \( \lor \), a consequence relation \( \vdash \) is a reflexive and transitive relation on \( W \) which is compatible with the classical connectives that are in \( C \), i.e. the following must always hold:

(1) If \( A, B \) are any two sentences in the classical fragment of \( \varepsilon \) such that \( A \rightarrow B \) is a classical tautology then \( A \vdash B \) holds, for any simultaneous substitutions, (from \( W \), \( A' \), \( B' \), of \( A, B \).)\n
(2) If \( A, B \) are any sentences of the classical fragment of \( \varepsilon \) and \( \vdash (A \rightarrow B) / A \rightarrow B \) is a valid rule of classical logic then \( A \vdash B \) imply \( A \vdash B \).

Clearly, for any semantics \( T, \vdash_T \) is a consequence relation on \( W \).

A theory of the languages is a pair \((\Delta, \Theta)\) of sets of sentences. \((\Delta, \Theta)\) is complete iff \( \Delta \cup \Theta = W \). Given a semantics \( T \), each \( s \in T \) defines a theory in a natural way, i.e.

\[ (\{A \mid s(A) = 1\}, \{A \mid s(A) = 0\}). \]

Given a consequence relation \( \vdash \) and a theory \((\Delta, \Theta)\), we say that \((\Delta, \Theta)\) is \( \vdash \) consistent iff for no \( A_i \in \Delta, B_i \in \Theta \) do we have \( \bigwedge A_i \vdash B_i \).

A logical system is simply a consequence relation \( \vdash \subseteq W \times W \) closed under substitution. We are particularly interested in logical systems that have some interesting intuitive meaning and are recursively enumerable.

Let \( T \) be a semantic and let an \( s \in T \), and a theory \((\Delta, \Theta)\) be given, we say that \( s \) is a model of \((\Delta, \Theta)\) iff for all \( A, A \in \Delta \) implies \( s(A) = 1 \), and \( A \notin \Theta \) implies \( s(A) = 0 \). If \( s(A) = 1, A \) is said to hold in the model \( s \), and if \( s(A) = 0, A \) is said to be false in the model. A semantics \( T \) is sound for a logical system \( \vdash \) iff \( \vdash \subseteq \vdash_T \). \( \vdash \) is said to be strongly complete for \( T \) if \( T \) is sound for \( \vdash \) and every \( \vdash \) consistent theory has a model in \( T \). \( \vdash \) is said to be (weakly) complete for \( T \) iff \( \vdash \vdash_T \).

39. WEAK ENTAILMENT SYSTEMS

1. GENERAL ENTAILMENT TYPE LOGICS

The structures have the form \((S, R, *, 0, D_t)\) where \( S \) is the set of possible worlds, \( R \subseteq S^3, * \) is a function in \( S^3, 0 \in S \) and \( D_t, \) for \( t \in S, \) is