CHARACTERIZATION OF THE EXPONENTIAL DISTRIBUTION USING LOWER MOMENTS OF ORDER STATISTICS*

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SUMMARY. Several characterizations of the exponential distribution in terms of the variances and the covariances of order statistics in a random sample of size n (n > 2) are made. Analogous characterizations hold for the positive exponential distribution.

KEY WORDS. Characterization, exponential distribution, order statistics.

1. INTRODUCTION. Rényi [8] has not only characterized the exponential distribution but also gave a representation for the exponential order statistics in a random sample of size n, which enables one to compute the moments of exponential order statistics. Epstein and Sobel [1] have given certain theorems which are helpful for computing the moments of exponential order statistics.

Several characterizations of the exponential distribution have been made notably by Ferguson [2], Ghurye [3], Tanis [11], Govindarajulu [6], and Rossberg [9, 10] using independence properties of certain functions of order statistics. In the following we shall characterize the exponential distribution using the relations among variances and covariances of exponential order statistics. All the results are given for the negative exponential distribution. Analogous results hold for the positive exponential distribution.

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2. NOTATION. Let $X_1, X_2, \ldots$, be independent nontrivial random variables each having the distribution function $F(x)$ with $F(0) = 0$. Also let $X_1, n \leq X_2, n \leq \ldots \leq X_n, n$, with $X_{0n} = 0$ and $X_{n+1, n} = \infty$, denote the ordered values of $X_1, \ldots, X_n$. Assume that $\mathbb{E}X_1^2 < \infty$. Let $U_1, U_2, \ldots$, be independent, each being uniformly distributed over the open interval $(0,1)$. Define

$$H(u) = \inf\{x \mid F(x) \geq u\}, \quad 0 < u < 1,$$

(2.1)

where $F$ is taken to be right continuous. Then, for $0 < u < 1$

$$H(u) \leq x \iff u \leq F(x),$$

(2.2)

and the distribution of $H(U_1), H(U_2), \ldots$, is the same as that of $X_1, X_2, \ldots$. Also, $H(U_{n,n}) = \max_{1 \leq k \leq n} H(U_k)$ has the same distribution as $X_{n,n}$, etc. Throughout, we assume that $F$ is absolutely continuous. That is, $H'(u)$ exists almost everywhere for $0 < u < 1$. Further, let

$$C_{i,j,k} = k!/i!j!, \quad I(u;k,m) = u^k(1-u)^m, \quad 0 < u < 1,$$

(2.3)

and

$$\mu^{(k)}_{i,n} = E(X_i^{k,n}) = C_{i-1,n-i,n} \int_0^\infty x^k I(F;i-1,n-1)dF(x)$$

(2.4)

for $i = 1, \ldots, n, k = 1, 2, \ldots$.

Then, we have the well-known recurrence relation (see, for instance, Govindaraju1u ([4], Theorem 4.1)):

$$(n-i)\mu^{(k)}_{i,n} + i\mu^{(k)}_{i+1,n} = n\mu^{(k)}_{i,n-1}, \quad k = 1, 2, \ldots, i = 1, 2, \ldots, n = i+1, i+2, \ldots$$

(2.5)

Let

$$C_{i,j,k,m} = m!/i! j! k! \quad \text{and} \quad I(u,v;i,k,m) = u^i(v-u)^k(1-v)^m$$

(2.6)

for $i,j,k,m \geq 0$ and $0 < u < v < 1$.

Also, let