CHAPTER X

THE STATISTICS OF NON-BOOLEAN
EVENT STRUCTURES

The Kochen and Specker theory of partial Boolean algebras leads to the resolution of the core problem of interpretation of quantum mechanics, the problem of hidden variables. To recapitulate: Quantum mechanics incorporates an algorithm for assigning probabilities to ranges of values of the physical magnitudes:

\[ p_w(a \in S) = \text{Tr}(W P_A(S)) \]

where \( W \) represents a statistical state of the theory, and \( P_A(S) \) is the projection operator onto the subspace in Hilbert space associated with the range \( S \) of the magnitude \( A \). The statistical states generate all possible (generalized) probability measures on the partial Boolean algebra of subspaces of Hilbert space. Joint probabilities

\[ p_w(a_1 \in S_1 \& a_2 \in S_2 \& \cdots \& a_n \in S_n) = \text{Tr}(W P_{A_1}(S_1)P_{A_2}(S_2)\cdots P_{A_n}(S_n)) \]

are defined only for compatible magnitudes \( A_1, A_2, \ldots, A_n \), and there are no dispersion-free statistical states. The problem of hidden variables concerns the possibility of representing the statistical states of quantum mechanics by measures on a classical probability space in such a way that the algebraic structure of the magnitudes of the theory is preserved. This is the problem of imbedding the partial algebra of magnitudes into a commutative algebra or, equivalently, the problem of imbedding the partial Boolean algebra of idempotent magnitudes (properties, propositions) into a Boolean algebra. The imbedding turns out to be impossible; there are no 2-valued homomorphisms on the partial Boolean algebra of idempotents of a quantum mechanical system, except in the case of a system associated with a 2-dimensional Hilbert space. Thus, the transition from classical to quantum mechanics involves the generalization of the Boolean propositional or event structures of classical mechanics to a particular class of non-Boolean structures. This may be understood as a generalization of the classical notion of validity: The class of models over which

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validity is defined is extended to include partial Boolean algebras which are not imbeddable into Boolean algebras.

In a Boolean algebra $\mathcal{B}$, there is a one-one correspondence between atoms, ultrafilters, and 2-valued homomorphisms, essentially because an ultrafilter $\phi$ in $\mathcal{B}$ contains $a$ or $a'$, but not both, for every $a \in \mathcal{B}$. If $b$ is an atom, either $a$ or $a'$ is above $b$, i.e. either $a \leq b$ or $a \leq b'$ for every $a \in \mathcal{B}$ (but not both, or else $b = 0$). Hence, there can be one and only one ultrafilter containing an atom. A 2-valued homomorphism is definable on $\mathcal{B}$ by mapping each element $a \in \mathcal{B}$ onto 1 or 0 according to whether $a$ is or is not a member of the ultrafilter $\phi$.

In a partial Boolean algebra that is not imbeddable in a Boolean algebra, the one-one correspondence between atomic events, ultrafilters, and 2-valued homomorphisms no longer holds. The partial Boolean algebra may be regarded as a partially ordered system, so the notion of a filter (and hence an ultrafilter as a maximal filter) is still well-defined. But it is no longer the case that if $\phi$ is an ultrafilter, then or each $a \in \mathcal{A}$ either $a \in \phi$ or $a' \in \phi$, and hence ultrafilters do not define 2-valued homomorphisms on $\mathcal{A}$. This is because ultrafilters (maximal filters) are no longer prime filters. A filter $\phi$ is prime if it is proper (i.e. a proper subset of $\mathcal{A}$), and if $a \lor b \in \phi$ only if either $a \in \phi$ or $b \in \phi$. Every ultrafilter $\phi$ in a partial Boolean algebra contains the unit, and hence contains $a \lor a'$ for every $a \in \mathcal{A}$. But if $\phi$ is an ultrafilter in the maximal Boolean sub-algebra $\mathcal{B} \subset \mathcal{A}$, then neither $a$ nor $a'$ will belong to $\phi$ if $a$ and $a'$ are outside $\mathcal{B}$, i.e. incompatible with the elements contained in $\phi$. An atom in $\mathcal{A}$ will correspond to an ultrafilter, but not to a prime filter, and hence will not define a 2-valued homomorphism on $\mathcal{A}$.

The Stone isomorphism maps every element in a Boolean algebra onto the set of ultrafilters containing the element. Thus, a measure on a classical probability space $X$ may be interpreted as a measure over ultrafilters or atomic events in a Boolean algebra $\mathcal{B}$, the points $x \in X$ corresponding to ultrafilters in $\mathcal{B}$ and the singleton subsets $\{x\}$ in $\mathcal{F}$ corresponding to atomic events. The probability of an event $a$ may be understood as the measure of the set of ultrafilters containing $a$, or the measure of the set of atomic events that can occur together with the event $a$:

$$p(a) = \mu(\phi_a).$$

The conditional probability of $a$ given $b$, $p(a | b)$, is the measure of the