Chapter 9

Bose–Einstein gases

This chapter discusses the properties of an ideal Bose–Einstein (BE) gas, which without any interactions nevertheless shows a remarkable phase transition, the 'Bose–Einstein condensation'. This property is relevant to liquid $^4$He. But the principal application of BE statistics is to the 'phoney' boson gases, photons and phonons.

9.1 Properties of an ideal Bose–Einstein gas

9.1.1 The Bose–Einstein distribution

In Chapter 5 (equations (5.11) and (5.13)) we derived the form of the distribution function, i.e. the number of particles per state of energy $\varepsilon$ in thermal equilibrium. For a gas of ideal bosons the distribution is

$$f_{BE}(\varepsilon) = 1/[B \exp(\varepsilon/k_B T) - 1].$$

(9.1)

For clarity the subscript BE will be omitted in the rest of this chapter. The parameter $B$ is to be determined from the number conditions $\sum n_i = N$, which caused its appearance in the first place (see again Chapter 5). We shall discuss this number condition in terms of $B$ for the boson gas, but it is entirely equivalent to the use of $\alpha$ or $\mu$ from the identities:

$$B = \exp(-\alpha) = \exp(-\mu/k_B T).$$

Before coming to the determination of $B$, the main task of this section, we can observe some of its properties just from inspection of the distribution (9.1). To be specific, let us
measure the one-particle energies $\varepsilon$ using the ground state as the zero of energy, i.e. $\varepsilon_0 = 0$. Now to make physical sense, we know that for all $\varepsilon$ the distribution $f(\varepsilon)$ must be positive. Because of the boson $-1$ in the denominator, this requires $B > 1$. If $B$ were negative, then at least the ground state occupation would be negative! Furthermore, if one were to suppose $B = 1$, then the ground state occupation would be infinite, a possibility in a ‘friendly’ boson system since there is no exclusion principle. Hence we must have $B > 1$ to describe the finite number $N$ of gas particles.

The value of $B$ will be found to vary with density and temperature, as did $\mu$ in the discussion of FD statistics of $\alpha$ in the case of MB statistics. We can recognize from equation (9.1) that a simplification to the MB limit will occur when $B \gg 1$. Under these circumstances the $-1$ in the denominator will become insignificant for all values of $\varepsilon$, even for the ground state. Hence equation (9.1) could be replaced by the simple

$$f_{\text{MB}}(\varepsilon) = (1/B) \exp (-\varepsilon/k_B T).$$

The obvious approach to determining $B$, and the one used with success in the MB and FD cases, is to use the density of states approximation, and to replace the sum over all states by an integration over $k$ or $\varepsilon$. So we enumerate the states by the usual function:

$$g(k)dk = V/(2\pi)^3 \cdot 4\pi k^2 \cdot dk \cdot G. \quad (9.2) \text{ and (4.5)}$$

We shall for simplicity consider spin 0 bosons for which the spin factor $G = 1$. Making the substitution into equation (9.1) of $\varepsilon = \hbar^2 k^2/2M$, as is appropriate for a gas of particles of mass $M$, we obtain:

$$N = \sum_i n_i$$
$$= \sum_i g_i f_i$$
$$= \int_0^\infty g(k) f(k) dk$$
$$= V/(2\pi)^3 \cdot 4\pi \int_0^\infty \frac{k^2dk}{[B \exp (\hbar^2 k^2/2Mk_BT) - 1]}$$

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