Fuzzy logic and categories of fuzzy sets

O. Wyler

Introduction

This paper deals with three topics:
1. Fuzzy logic,
2. Categories of fuzzy sets,
3. Logic of fuzzy subsets.

While much of its contents can be found in my book [17] and in the existing literature, it also includes new results and a large amount of unpublished folklore. There is also a section discussing a formal language for fuzzy logic, with interpretations of formulas. Thus I believe that this coherent and not too technical survey of fuzzy logic and categories of fuzzy sets is useful. I have tried to make the paper reasonably self-contained, except that I use the basic language of categories freely. Proofs in Sections 1 and 2 are mostly omitted; they are usually straightforward or can be found in the given references.

When L. ZADEH [18] introduced fuzzy sets, he regarded fuzzy sets essentially as “crisp” sets with a \([0, 1]\)-valued membership degree function. Membership degrees were soon perceived as truth-values, and this called for a fuzzy logic, with truth-values in the unit interval. ZADEH used Łukasiewicz logic for propositional connectives, without saying why, and most users of fuzzy logic have followed his example. There have been claims in the literature that Łukasiewicz logic must be used in certain applications, but these claims do not stand up to scrutiny.

As J. GOGUEN [6] soon pointed out, there is no mathematical need to use the real unit interval as set of truth-values, or to use Łukasiewicz logic. For technical reasons, truth-values must form a complete lattice, and preferably a complete Heyting algebra. This puts intuitionistic logic at our disposal, but it does not exclude non-intuitionistic propositional connectives and logics. We note that every order-complete chain, and in particular the real unit interval \([0, 1]\), is a complete Heyting algebra. Open sets of a topological space form a complete...
Heyting algebra, and every finite Heyting algebra is of this type, up to an isomorphism of Heyting algebras.

In some respects, logic and set theory are siamese twins. Set operations are based on logical connectives and quantifiers, and these connectives and quantifiers can be retrieved from the set operations. We cannot lay a foundation for fuzzy sets without fuzzy logic, and a language for fuzzy logic is based on a category of fuzzy sets.

Everybody working with fuzzy sets seems to agree that fuzzy sets are crisp sets with additional structure, and that this additional structure includes a degree of membership function with values in a complete lattice $H$. For the categorically minded, fuzzy sets are the objects of a category, and this category should be as set-like as possible. J. GOGUEN [6] was the first, but by no means the last author to present such a category, and it seems likely that there is no single category of fuzzy sets which satisfies all needs. There are basic questions which have not yet been answered to everybody's satisfaction. Two of these questions ask: how fuzzy should things be? Should equality be fuzzy or crisp? And should morphisms be crisp maps, extensional maps, or fuzzy functions? Another important question: what should the underlying fuzzy logic of our set theory be?

We do not try to answer the first two of these three questions; thus Section 2 presents six categories of fuzzy sets, each equipped with fuzzy logic, from which the reader can choose. On the other hand, we do not leave the choice of logics open; we base our categories on intuitionistic logic. This needs some discussion.

When we choose a lattice $H$ of truth-values for membership degrees, then meets and joins in $H$ define standard conjunction and disjunction. If $H$ is complete, then infima and suprema in $H$ define universal and existential quantifiers which generalize standard conjunction and disjunction. By basic principles, implication should be right adjoint to conjunction, and this right adjoint exists if $H$ is a Heyting algebra. The standard logic thus obtained is intuitionistic, it is always there, and it agrees with the basic categorical constructions. Other logics can also be obtained and used, depending on the choice of $H$, but basing a category of fuzzy sets on a non-intuitionistic logic seems to lead to complications. There may be gains justifying these complications, but I do not see them at this time. Another point is that categories with intuitionistic fuzzy logic may serve as models for categories with a non-standard fuzzy logic.

A sufficiently set-like category has an internal logic. This logic is intuitionistic when it exists, but it may not be what we want and need. For example, the internal logic of Goguen's category of $H$-valued fuzzy sets is always crisp, i.e. classical with just two truth-values. L.N. STOUT [16] has shown a way out of this seeming contradiction. A generalization of Stout's theory, based on a notion of fuzzy subset, was obtained in [17]. For the six categories constructed in Section 2, we present this theory in Section 3.

In Section 2, we justify or motivate axioms for fuzzy sets and maps by translating them informally into a first-order language. Section 4 tries to un-