DOES $R$ PRIME IMPLY $M_R(R^2)$ IS SIMPLE?  

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ABSTRACT

In [1] Maxson and van der Walt prove that when $M_R(R^2)$ is simple, then $R$ must be prime. In this paper we present a class of examples which proves the converse need not be true. That is, $R$ prime does not imply $M_R(R^2)$ is simple.

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Introduction

Let $R$ be a ring and $M_R(R^2)$ the set of all maps on $R^2 = R \oplus R$ with the property that scalar multiples can be pulled out. That is, $M_R(R^2) = \{f \mid f : R^2 \to R^2 \text{ and } f([\frac{a}{b}, \frac{c}{d}]) = ([\frac{a}{b}]r) \text{ for all } [\frac{a}{b}] \in R^2 \text{ and } r \in R\}$. Under function composition and pointwise addition, $M_R(R^2)$ forms a right near-ring. In [1] it was shown that if $M_R(R^2)$ is simple, then $R$ must be a prime ring, and in the case that $R$ was known to be commutative, $R$ prime implies $M_R(R^2)$ simple. It was left open whether the converse was true. That is whether or not $R$ prime implies $M_R(R^2)$ is simple.

We first recall that a ring $R$ is prime if and only if for all ideals $I$ and $J$ of $R$, $IJ = \{0\}$ or $JI = \{0\}$ implies either $I$ or $J$ is $\{0\}$. Herein we shall exhibit a class of primitive (and hence prime) rings for which $M_R(R^2)$ is not simple, thereby settling the question negatively.

Throughout this work we shall denote the image space of a homomorphism $c$ by $\mathcal{S}c$. The next lemma is basic.

Lemma 1. Let $V$ be an infinite–dimensional vector space over a division ring $D$ and let $c_1, c_2 \in \text{Hom}_D(V, V)$ such that $\dim(\mathcal{S}c_1)$ and $\dim(\mathcal{S}c_2)$ are strictly less then the dimension of $V$. Then $\dim(V/(\ker c_1 \cap \ker c_2)) = [V : \ker c_1 \cap \ker c_2]$ is strictly less then the dimension of $V$. In particular, when $\dim(\mathcal{S}c_1) < \infty$, and $\dim(\mathcal{S}c_2) < \infty$, then $[V : \ker c_1 \cap \ker c_2] < \infty$.

Proof: Let $I_1 = \dim \mathcal{S}c_1$, and $I_2 = \dim \mathcal{S}c_2$. Without loss of generality there exist linearly independent sets $X_1$ and $X_2$ in $V$ with the following properties:(e.g. $X_1$ and $X_2$ are selected such that $c_1(X_1)$ is a basis for $\mathcal{S}c_1$, and $c_2(X_1 \cup X_2)$ contains a basis for $\mathcal{S}c_2$.)

(i) $|X_1| = I_1, |X_2| \leq I_2$,

(ii) $\mathcal{S}c_1 = c_1(\text{span} X_1)$,

Footnote: This work is part of the author’s doctoral dissertation at Texas A & M University.

Y. Fong et al. (eds.), Near-Rings and Near-Fields, 53–56.
(iii) \( \exists c_2 = c_2(\text{span } (X_1 \cup X_2)) \),
(iv) \( X_2 \subset \ker c_1 \),
(v) \( c_1(X_1) \) and \( c_2(X_2) \) are linearly independent sets, and
(vi) \( X_1 \cup X_2 \) is a linearly independent set.

We shall show that for all \( v \in V \), \( v \) can be written as a sum \( v_1 + k \) where \( v_1 \in \text{span } (X_1 \cup X_2) \) and \( k \in \ker c_1 \cap \ker c_2 \). Since \( I_1 < \dim V \) and \( I_2 < \dim V \) this implies
\[
\dim(\text{span } (X_1 \cup X_2)) \leq I_1 + I_2 < V
\]
which will yield the result.

By properties (ii) and (v), \( c_1(X_1) \) is a basis for \( \Im c_1 \). By (iii), (v), and (vi), \( c_2(X_2) \) can be extended to a basis for \( \Im c_2 \) in such a way that there exists a set \( X_3 \subset X_1 \) such that for all \( v \in V \), \( c_2(v) = \sum a_i c_2(y_i) + \sum b_j c_2(z_j) \) where \( a_i, b_j \in D, y_i \in X_2, z_j \in X_3 \subset X_1 \), \( \{ c_2(X_2) \cup c_2(X_3) \} \) is a basis for \( \Im c_2 \), and \( X_1 \setminus X_3 \subset \ker c_2 \).

Define \( c'_1 : \Im c_1 \to \text{span } (X_1) \) by \( c'_1(v) = \sum a_i x_i \) if \( v = \sum a_i c_1(x_i), a_i \in D, x_i \in X_1 \). Similarly, define \( c'_2 : \Im c_2 \to \text{span } (X_2 \cup X_3) \) by
\[
c'_2(v) = \sum a_i y_i + \sum b_j z_j
\]
if
\[
v = \sum a_i c_2(y_i) + \sum b_j c_2(z_j), \quad a_i, b_j \in D, y_i \in X_2, z_j \in X_3.
\]

Let \( v \in V \). Let \( k = v - c'_1 c_1(v) - c'_2 (c_2(v - c'_1 c_1(v))) \). We claim \( k \in \ker c_1 \cap \ker c_2 \).
By construction, \( c_2(c'_2 c_2(x)) = c_2(x) \) for all \( x \in V \) so
\[
c_2(k) = c_2(v - c'_1 c_1(v)) - c_2(v - c'_1 c_1(v)) = 0.
\]
Also
\[
c_1(k) = c_1(v) - c_1(v) - c_1 c'_2 (c_2(v) - c'_1 c_1(v))
    = -c_1 c'_2 (c_2(v) - c'_1 c_1(v)).
\]
Claim:
\[
c'_2(c_2(v) - c'_1 c_1(v)) \in \text{span } X_2
\]
which implies
\[
-c_1 c'_2 (c_2(v) - c'_1 c_1(v)) = 0
\]
by property (iv).
Noting that \( c'_1 c_1(v) = \sum b_j z_j + \sum d_k w_k, \ b_j, c_k \in D, z_j \in X_3, w_k \in X_1 \setminus X_3 \),
then
\[
c_2 c'_1 c_1(v) = \sum b_j c_2(z_j)
\]
and
\[
c_2(v) - c_2 c'_1 c_1(v) = \sum a_i c_2(y_i), \ a_i \in D, y_i \in X_2
\]