THE IDENTITY \((x^2)^2 = W(x)x^3\) IN BARIC ALGEBRAS

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Abstract In this paper we study the identity \((x^2)^2 = w(x)x^3\) and we prove that this identity characterizes power-associative Bernstein algebras of order 2, \(A = Ke \oplus U \oplus V_2\) with \(v^3 = 0\) for every \(v \in V_2\). Moreover, we study the generalized Etherington’s ideals of \(A\).

1. Introduction

In what follows, \(K\) is an infinite field of characteristic not 2 and \(A\) is a commutative non necessarily associative algebra over \(K\).

We recall that \(A\) is a power-associative algebra if every subalgebra generated by only one element is associative. If \(w : A \rightarrow K\) is a nonzero algebra homomorphism, then the ordered pair \((A, w)\) is called a baric algebra and \(w\) its weight homomorphism. If the baric algebra \((A, w)\) satisfies the identity \(x^{n+2} = (w(x)x)^{n+1}\), it is called a Bernstein algebra of order \(n\) where \(n\) is the minimum integer for which the identity holds and \(x^{[n]} = x, \ldots, x^{[k+1]} = x^{[k]}x^{[k]}, k \geq 1\) are the plenary powers of \(x\). For references, see [7]. If the baric algebra \((A, w)\) satisfies the equation \(e^n + \gamma_1 w(x)e^{n-1} + \ldots + \gamma_{r-1} w(x)^{r-1} x = 0\) (train equation), it is called a train algebra of rank \(r\), where \(r\) is the minimum integer for which the above identity holds, \(\gamma_1, \ldots, \gamma_{r-1}\) are fixed elements in \(K\) and \(x^1 = x, \ldots, x^{k+1} = x^k x\) are the principal powers of \(x\). The baric algebra \((A, w)\) is a special train algebra if \(\text{Ker}(w)^k\) is an ideal of \(A\) for every \(k \in \mathbb{N}\) and \(\text{Ker}(w)\) is nilpotent. Moreover, every special train algebra is a train algebra, (for details see [8]).

It is well known that for Bernstein algebras \(A\) of order 2, the homomorphism \(w\) is uniquely determined and the set of idempotents is given by \(I_p(A) = \{(x^2)^2 / x \in A, w(x) = 1\}\). Moreover, \(A\) splits as the direct sum relative to one idempotent \(e \neq 0\), \(A = Ke \oplus \text{Ker}(w) = Ke \oplus U \oplus V_2\), where \(U = \{x \in \text{Ker}(w)/ex = \frac{1}{2} x\}\) and \(V_2 = \{x \in \text{Ker}(w)/e(ex) = 0\}\) and \(U^2 \subseteq V_2\).

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2. The identity $(x^2)^2 = w(x)x^3$

The idea of studying the identity

$$(x^2)^2 = w(x)x^3 \quad (1)$$

in baric algebras arise of the study of power-associative Bernstein algebras of order 2, i.e. power-associative, baric algebras satisfying $((x^2)^2)^2 = w(x^4)(x^2)^2$.

In [6], it is proved the following result that characterize power-associative Bernstein algebras of order 2.

**Theorem 2.1** For a baric algebra $(A, w)$, of arbitrary dimension, the following conditions are equivalent:

1. $(A, w)$ is a power-associative Bernstein algebra of order 2.
2. There exist a decomposition of $A$ relative to an idempotent $e \neq 0$, $A = Ke \oplus U \oplus V_2$ where $\text{Ker}(w) = U \oplus V_2$, $U = \{ x \in \text{Ker}(w)/ex = \frac{1}{2}x \}$, $V_2 = \{ x \in \text{Ker}(w)/ex = 0 \}$, $U^2 \subseteq V_2$, $V_2^2 \subseteq V_2$, $UV_2 \subseteq U$ and every element $u \in U$ and $v \in V_2$ satisfy the relations: $u^3 = 0$, $v^4 = (v^2)^2 = 0$, $uv^2 = 2v(uv)$, $u^2v = 2u(uv)$.

Moreover, for every $u, u' \in U, v, v' \in V$ we have

$$(u^2)^2 = 0, \quad (2)$$

$$u(vv') = (uv)v' + (uv')v, \quad (3)$$

$$v(uu') = (vu)u' + (vu')u, \quad (4)$$

$$4(uv)^2 + u^2v^2 = 2v(uv^2). \quad (5)$$

On the other hand, in [1], it is proved that algebras satisfying the identity (1) always have idempotent elements and every linear form $w : A \to K$ is also a multiplicative map. Moreover, $A$ admits a Peirce decomposition $A = Ke \oplus N_\frac{1}{2} \oplus N_0$, where $N_i = \{ x \in \text{Ker}(w)/ex = ix \}$, $i = 0, \frac{1}{2}$ and $N_\frac{1}{2} \subseteq N_0$, $N_\frac{1}{2}N_0 \subseteq N_\frac{1}{2}$, $N_0^2 \subseteq N_0$ and for every $u \in N_\frac{1}{2}$, $v \in N_0$: $u^3 = 0$, $v^3 = 0$, $uv^2 = 2v(uv)$, $uv^2 = 2u(uv)$.

Thus, we have

**Proposition 2.2** Every baric algebra satisfying the identity (1) is a power-associative Bernstein algebra of order 2.

**Remark** The converse of Proposition 2.2 is not true. For example, $A = Ke \oplus V_2$, $V_2 = \langle v_1, v_2, v_3 \rangle$ and multiplication table $e^2 = e$, $v_1^2 = v_2$, $v_1v_2 = v_3$, all other products being zero is a Bernstein algebra of order 2. Moreover, it is Jordan, so, it is a power-associative algebra and if $x = e + v_1 + v_2 + v_3$ then $(x^2)^2 = e$ and $w(x)x^3 = e + v_3$. 