Chapter 4

General technique for constructing linear RA for linear problems in Hilbert space

Speaking of linear ill-posed problems in Hilbert spaces, we henceforth bear in mind the two following particular problems:

Problem 1. Solve the operator equation

\[ Az = u, \quad z \in Z, \quad u \in U \]  \hspace{1cm} (4.1)

with a closed and, generally speaking, uninvertible operator \( A \). The domain of definition of \( A D_A \) is dense in \( Z \).

Problem 2. Calculate the values of closed unbounded operator \( A : Z \to U \)

\hspace{1cm} (4.2)

with a dense domain \( D_A \).

These two problems are closely related. The first of them may be formally reduced to the second if the inverse operator \( A^{-1} \) is appropriately defined. However, such an approach is not always fruitful due to different reasons. It proves more convenient to construct regularizing families \( R_\alpha \) independently for each of the Problems 1, 2. It is possible, for instance, to reduce the problem of differentiating a function (Problem 2) to the solution of integral equation of the first kind (Problem 1), but such a transfer...
is obviously unnecessary for regularizing the problem of differentiating. For the above reason we do not discuss the formal schemes universal for both problems, but rather investigate these problems separately.

### 4.1 General scheme for constructing RA for linear problems with completely continuous operator

Suppose that operator $A$ in Problem 1 is completely continuous, and $Z, U$ are separable Hilbert spaces.

Denote the spectrum of operator $A^*A$ by $S(A^*A)$. It consists of eigenvalues of operator $A^*A$ and zero point, $\lambda = 0$ not necessarily being the eigenvalue of $A^*A$. Since $A^*A$ is positively defined, its spectrum $S(A^*A)$ belongs to the positive half of the axis of real numbers in a complex plane and consists of no more than a countable set of points $\lambda_i$. Let $e_i$ be the eigenfunction of operator $A^*A$ corresponding to the eigenvalue $\lambda_i$. The set of $e_i$ constitutes an orthonormalized sequence in $Z$. Now we define a function of the operator $A^*A$. Let $\chi(\lambda)$ be a complex bounded function defined at $\lambda \in S(A^*A)$. The operator $\chi(A^*A)$ is defined by equation

$$
\chi(A^*A)f = \sum_{\lambda_i \neq 0} \chi(\lambda_i)(f, e_i)e_i + \sum_{\varepsilon, \in \text{Ker}(A^*A)} \chi(0)(f, e_i)e_i.
$$

In Eq. (4.3) we have separated summing over the eigenfunctions from Ker($A^*A$) (if such functions exist). Eq. (4.3) defines a linear bounded operator, its norm being

$$
\|\chi(A^*A)\| = \sup_{\lambda \in S(A^*A)} |\chi(\lambda)|.
$$

To solve the problem given by Eq. (4.1) approximately, we should (in accordance with general concept described in Chapter 1) construct the family of bounded operators $R_o$ which yields a pointwise approximation of mapping $G = A^{-1}$ generated by Eq. (4.1). In general, $A^{-1}$ is a one-to-many mapping since we do not assume that Ker$A = \emptyset$. Henceforth (if the opposite is not explicitly stated) we assume that $G = A^{-1}$ is a one-to-one section of the above-mentioned mapping such that $Gu = \arg \min_{A^2u} ||z||$.

The value $Gu$ given by the latter one-to-one mapping is also referred to as a normal (least norm) solution of Eq. (4.1). It is easy to see that a normal