Chapter 12

Quasi-Homogeneous Dynamics in the Plane

PETER GROSS
Department of Mathematics and Computer Science, University of Aalborg

BJARNE SLOTH JENSEN
Institute of Economics, Copenhagen Business School

Introduction

The principal aim of this chapter is to generalize the theory in chapter 10 to a wider class of differential equations, which are called quasihomogeneous, [7]. In section 12.1, we characterize the class of real quasihomogeneous functions on \( \mathbb{R}^2 \). Then we consider quasihomogeneous differential equations and examine some general aspects of the solutions. Section 12.3 discusses the stability properties of quasihomogeneous dynamics within the framework presented in the addendum. Concluding comments are offered in section 12.4.

12.1 The class of quasi-homogeneous functions

In accordance with the concept presented in ([7], p. 5), we introduce:

Definition 1. A nonvanishing \( C^0 \)-function \( f : \mathbb{R}^2 \to \mathbb{R} \) is quasihomogeneous, if there exist real numbers \( \beta_1, \beta_2, \tilde{m}; \beta_1, \beta_2 \neq 0 \), such that

\[
\forall (x, y) \in \mathbb{R}^2 \forall \mu \in \mathbb{R}_+ : f(\mu^{\beta_1} x, \mu^{\beta_2} y) = \mu^{\tilde{m}} f(x, y). \tag{12.1}
\]

The pair \((\beta_1, \beta_2)\) are called the weights of \( f \), and \( \tilde{m} \) the associated degree of quasi-homogeneity. Clearly, if \((\beta_1, \beta_2)\) are weights for \( f \) with associated degree \( \tilde{m} \), then \((A \beta_1, A \beta_2)\) are weights for \( f \) with associated degree \( A \tilde{m} \), \( A \in \mathbb{R} \setminus \{0\} \). To remove such redundancy, we state

Lemma 1. Given a quasihomogeneous \( C^0 \)-function \( f : \mathbb{R}^2 \to \mathbb{R} \). Then there exist a \( \beta \neq 0 \) and an \( m \in \mathbb{R} \), such that \((1, \beta)\) are the weights and \( m \) the degree of \( f \).

Proof. Let \((\beta_1, \beta_2)\) be arbitrary weights for \( f \), and \( \tilde{m} \) the associated degree. Putting \( \lambda = \mu^{\beta_1}, \beta = \frac{\beta_2}{\beta_1}, m = \frac{\tilde{m}}{\beta_1}, \) we have for all \( \lambda > 0 \), cf. (12.1)

\[
f(\lambda x, \lambda y) = f(\mu^{\beta_1} x, \mu^{\beta_2} y) = \mu^{\tilde{m}} f(x, y) = \lambda^{\beta_1} f(x, y) = \lambda^m f(x, y), \tag{12.2}
\]

which proves that \((1, \beta)\) and \( m \) are weights and associated degree of \( f \). \( \square \)
Henceforth, by the weight $\beta$ and the degree $m$ of a quasihomogeneous function $f$, we always mean the unique values chosen in Lemma 1.

Observe that any homogeneous $C^0$-function is quasihomogeneous with weight $\beta = 1$ and degree $m$ equal to the degree of homogeneity, i.e., the class of quasihomogeneous functions contains the class of homogeneous functions.

Remark 12.1. It follows immediately from Definition 1, Lemma 1 and (12.2) with $\lambda = 1/x$ that a nonvanishing $C^0$-function $f : \mathbb{R}^2 \to \mathbb{R}$ is quasihomogeneous with weight $\beta$ and degree $m$, if and only if it can be written as

$$f(x, y) = x^m f(1, y/x^\beta) = x^m f(1, r^\beta), \quad r = y^{1/\beta}/x, \quad x \neq 0. \quad (12.3)$$

With differentiability, a very useful characterization of the quasihomogeneous functions is given by the following generalization of Euler's relation:

**Theorem 1.** Assume $f \in C^1(\mathbb{R}^2)$. Then $f$ is quasihomogeneous of weight $\beta \neq 0$ and degree $m$, if and only if $f$ satisfies the relation

$$x \frac{\partial f}{\partial x}(x, y) + \beta y \frac{\partial f}{\partial y}(x, y) = m \cdot f(x, y), \quad (x, y) \in \mathbb{R}^2. \quad (12.4)$$

**Proof.** If $f$ is quasihomogeneous, then by (12.2)

$$f(\lambda x, \lambda^\beta y) = \lambda^m f(x, y), \quad \lambda > 0, \quad (x, y) \in \mathbb{R}^2. \quad (12.5)$$

By the chain rule, (12.5) is differentiable in $\lambda$, and differentiation yields

$$x \frac{\partial f}{\partial x}(\lambda x, \lambda^\beta y) + \beta \lambda^{\beta-1} y \frac{\partial f}{\partial y}(\lambda x, \lambda^\beta y) = m \lambda^{m-1} f(x, y), \quad (12.6)$$

and putting $\lambda = 1$, we get (12.4).

Conversely, assume $f$ satisfies (12.4), and consider an arbitrary point $(x_0, y_0) \in \mathbb{R}^2$. Define the $C^1$-function $j : \mathbb{R}^2 \to \mathbb{R}$ by

$$j(\lambda) = f(\lambda x_0, \lambda^\beta y_0), \quad \lambda \in \mathbb{R}_+. \quad (12.7)$$

Applying (12.4), we have

$$j'(\lambda) = \frac{1}{\lambda} \left[ (\lambda x_0) \frac{\partial f}{\partial x}(\lambda x_0, \lambda^\beta y_0) + \beta (\lambda^\beta y_0) \frac{\partial f}{\partial y}(\lambda x_0, \lambda^\beta y_0) \right]$$

$$= \frac{m}{\lambda} f(\lambda x_0, \lambda^\beta y_0) = \frac{m}{\lambda} j(\lambda), \quad \lambda \in \mathbb{R}_+, \quad (12.8)$$

such that $z = j(\lambda)$ is a solution to the simple ODE

$$dz/d\lambda - (m/\lambda)z = 0. \quad (12.9)$$