POINCARÉ INEQUALITIES IN $L^1$-NORM FOR THE SPHERE
AND A STRONG ISOPERIMETRIC INEQUALITY IN $\mathbb{R}^n$

BENT FUGLEDE

University of Copenhagen, Mathematics Institute,
Universitetsparken 5, 2100 Copenhagen, Denmark

ABSTRACT. For real-valued functions on the unit sphere in $\mathbb{R}^n$ with mean-value 0
the $L^1$-norm is estimated from above by a best possible constant times the Dirichlet
norm. A dual version of this inequality involves the expansion of bounded functions
in a series of spherical harmonics. A similar pair of dual inequalities leads to a lower
estimate of the isoperimetric deficit of nearly spherical, convex domains $K$ in $\mathbb{R}^n$
in terms of the asymmetry of $K$, which is essentially the volume of the symmetric
difference of $K$ and a suitable ball.

Introduction

The inspiration to the present work came from the inequalities (2) and (5) below,
obtained recently by Hall, Hayman, and Weitsman [HH], [HHW]. We shall prove (or
reprove) all the inequalities (1) through (5) and address the problem of obtaining
higher dimensional versions of them. This extension requires a different approach
from that of [HH] which was based on subordination theory from complex analysis.

If $a_k, b_k$ are the Fourier coefficients of a measurable function $f$ on $\mathbb{R}$ (mod $2\pi$)
with values between $-1$ and 1 then

$$\frac{1}{2} \sum_{k=1}^{\infty} \frac{a_k^2 + b_k^2}{k^2} \leq \frac{\pi^2}{12},$$

$$\frac{1}{2} \sum_{k=2}^{\infty} \frac{a_k^2 + b_k^2}{k^2 - 1} \leq \frac{4}{\pi} - 1. \tag{2}$$

Both inequalities are sharp. The equality sign prevails in (1) e.g. for $f(\theta) = \text{sgn}(\cos \theta)$, and in (2) e.g. for $f(\theta) = \text{sgn}(\cos 2\theta)$. (See also below.) As pointed
out by the referee, (1) can be established by subordination, like (2) in [HHH].

Each of the inequalities (1), (2) has a dual version in terms of $2\pi$-periodic, abso-
lutely continuous real-valued functions $u$:

$$\left(\int_{-\pi}^{\pi} |u(\theta)| \frac{d\theta}{2\pi}\right)^2 \leq \frac{\pi^2}{12} \int_{-\pi}^{\pi} u'(\theta)^2 \frac{d\theta}{2\pi},$$

$$\left(\int_{-\pi}^{\pi} |u(\theta)| \frac{d\theta}{2\pi}\right)^2 \leq \left(\frac{4}{\pi} - 1\right) \int_{-\pi}^{\pi} (u'(\theta)^2 - u(\theta)^2) \frac{d\theta}{2\pi}. \tag{4}$$
where it is assumed in (3) that $\int_{-\pi}^{\pi} u(\theta) \, d\theta = 0$, and in (4) that the Fourier coefficients of $u$ of orders $k = 0$ and $k = 1$ are 0.

The sign of equality in (3) occurs for

$$u(\theta) = \frac{1}{2} ((\pi/2)^2 - \theta^2) \quad \text{for} \quad -\pi/2 \leq \theta \leq \pi/2,$$

continued so as to be $\pi$-antiperiodic, i.e. $u(\theta + \pi) = -u(\theta)$. The sign of equality in (4) holds for

$$u(\theta) = \sqrt{2} \cos \theta - 1 \quad \text{for} \quad -\pi/4 \leq \theta \leq \pi/4,$$

continued so as to be $(\pi/2)$-antiperiodic. In each of the inequalities (1) through (4) the equality sign holds only for the translates of the particular function $f$ stated above, resp. only for the constant multiples of translates of the above particular function $u$ (see Section 1 as to (1) and (3), and Section 2, or in detail [F3], as to (2) and (4)). In the case of (2) this uniqueness does not seem to follow from [HH].

Our primary aim in studying the inequalities (1) through (4), and especially their higher dimensional versions, is the determination of the best possible constants and of the functions for which the equality sign holds. If we would be content with bigger constants, the inequalities (1) through (4) would follow immediately from the corresponding, much simpler inequalities in which the $L^2$-norm of $f$ and $u$ enters in place of the $L^\infty$-norm of $f$ (implicit in (1) and (2)) and the $L^1$-norm of $u$ (on the left of (3) and (4)). For example, the $L^2$ analogue of (3) is known as Wirtinger's inequality (with the best possible constant 1 in place of $\pi^2/12$ in (3)). And the $L^2$ analogue of the $n$-dimensional version of (3) is the Poincaré inequality for the unit sphere in $\mathbb{R}^n$. For other Poincaré type inequalities see [MPF, p. 96 ff.].

The inequality (4) is implicitly contained in [HHW], where the classical isoperimetric inequality for convex bodies $K$ in the plane, with area $A(K) = A$ and perimeter $L$, was strengthened as follows by application of (2):

$$L^2 \geq 4\pi A \left( 1 + \frac{\pi}{4 - \pi} \alpha^2 + O(\alpha^3) \right).$$  \hspace{1cm} (5)

Here $\alpha$ denotes the following measure of the “average” deviation of $K$ from circular shape:

$$\alpha = \min_{x \in \mathbb{R}^2} \frac{A(K \setminus B(x, K))}{A(K)},$$

where $B(x, K)$ is the disk centred at $x$ and having the same area as $K$. This quantity $\alpha$ was proposed by L. E. Fraenkel and is known as the asymmetry of $K$. The constant $\pi/(4 - \pi)$ in (5) was shown in [HHW] to be best possible, briefly because it is the reciprocal of the best possible constant $4/\pi - 1$ in (2) and (4).

In Section 1 of the present article we establish (1) and (3) as the case $n = 2$ of suitable $n$-dimensional versions of them, including the explicit determination of the best possible constant $\kappa_n$ to replace $\pi^2/12$ if $n > 2$. In Section 2 we describe