Two Nonsmooth Approaches to Simultaneous Geometry and Topology Design of Trusses

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KEYWORDS: Optimal Design of Trusses, Nondifferentiable Optimization

1 The Problem

The optimization of the geometry and topology of structural layout has great impact on the performance of truss- and grillagelike structures and recent years have seen a revived interest in this area of structural optimization. Most of the work deals with analytical approaches to the subject while there is still a strong demand for efficient numerical methods. Our aim is to convince the reader that tools from nonsmooth optimization can be very helpful in this context. More precisely: we will show that with appropriate problem reformulations and by use of nonsmooth codes we can optimize trusses simultaneously with respect to geometry and topology.

In the following we consider the problem of finding the stiffest (maximum strength) truss which can carry a given load and which consists of perfect slender bars of homogeneous material. Let $m$ be the number of potential bars in the starting layout of the truss. The volumes $t_1, t_2, \ldots, t_m \geq 0$ of these bars will be design variables in the following (topology aspect). Further, we allow movements of the nodes in certain neighborhoods of their original positions $z_j$; the vector $y$ of these moves $y_j$ (relative to $z_j$) forms a second block of design variables (geometry aspect). Finally, the nodal displacements $x_j$ define a supplementary vector $x$ of unknowns. If $N$ is the number of nodes in the truss and $s$ the number of fixed nodal displacements (i.e., the number of unknowns with prescribed discrete homogeneous Dirichlet boundary condition), then $x$ and $y$ are of dimension

$$n = \dim \cdot N - s;$$

here $\dim$ is 2 for planar and 3 for spatial trusses. The number $m$ of potential bars will typically be very large in a topology approach compared to the number $N$ of nodes; usually one allows connections between each two of the nodes and consequently $m = N(N - 1)/2$.

The elastic equilibrium of such a truss subject to an external nodal force vector $f \in \mathbb{R}^n$ (the load) is given by

$$\sum_{i=1}^{m} t_i A_i (z + y) x = f.$$  \hfill (1)
Here
\[ A_i(z + y) = \frac{E_i}{l_i(z + y)^2} \gamma_i(z + y)(\gamma_i(z + y))^T \] (2)
is the symmetric and positive semidefinite \( n \times n \) dyadic stiffness matrix of the \( i \)-th bar, 
\( l_i(z + y) \) is the length of this bar, \( E_i \) its Young's modulus and \( \gamma_i(z + y) \) the \( n \)-vector of direction cosines such that \( \gamma_i^T(z + y)x \) is the bar elongation. In the following we will skip the fixed vector \( z \) of starting positions of the nodes and write \( A_i(y) \) for \( A_i(z + y) \) and, further, \( A_i \) for \( A_i(0) \). We would like to emphasize however that our approach below does not rely on the special form (2) for the stiffness matrices \( A_i \) and holds true for more general situations.

The compliance \( \frac{1}{2}f^T x \) is a measure for the stiffness of such a truss under the load \( f \), and the design goal in this paper is that of minimizing the compliance (which corresponds to maximizing stiffness or strength) of the truss. As variables we have the design vectors \( t \) and \( y \) of bar sizes and nodal positions, respectively, and the vector \( x \) of nodal displacements. Constraints are the equilibrium condition (1) and a maximal total volume \( V \) for the bars; further, only moves of the nodes within certain “balls” \( U_j \subset \mathbb{R}^{d_{nm}} \) are feasible. The \( U_j \) have, of course, to be chosen small enough such that the corresponding structures are well-defined trusses; if we want to fix the position of some node \( j \) then we put \( U_j := \{0_{\mathbb{R}^{3nm}}\} \). In mathematical terms this leads to the optimization problem in \((t,x,y) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n; \)

\[
\begin{align*}
(P)^y_{t,x} & \quad \min_{t,x,y} \frac{1}{2}f^T x \\
\text{subject to} & \quad \sum t_i A_i(y)x = f \\
& \quad 0 \leq t_i, \sum t_i \leq V, \\
& \quad y \in U := U_1 \times \ldots \times U_n.
\end{align*}
\]

We emphasize that we optimize in \((P)^y_{t,x} \) simultaneously with respect to topology \((t_i\)-variables) and to geometry \((y_j\)-variables). Even elements of shape optimization are addressed in our question, since the optimization may lead to a considerable modification of the starting shape; see the examples from Section 5. Unfortunately, \((P)^y_{t,x} \) is very hard to solve numerically. This is mainly due to the geometry variable \( y \), which appears even in the denominator of \( A_i \) (see (2)). Indeed, even for modest sizes of \( n \) and \( m \), standard software like SQP-variants failed in our test runs when trying to solve \((P)^y_{t,x} \).

In Sections 3 and 4 we will present two “equivalent” reformulations of \((P)^y_{t,x} \) which, at the cost of loss of differentiability, reduce \((P)^y_{t,x} \) to a nonsmooth displacement based problem \((D)_x \) for some dense underlying ground structure (Section 3) or, study \((P)^y_{t,x} \) as two-level problem (Section 4). By use of nonsmooth codes we can succesfully deal with these substitute problems. Section 5 will support this claim by a number of numerical examples.

Since nonsmooth optimization is fundamental for the following, some needed basic facts are summarized in the next section.