10. THE WAITING PROBLEM *

§1. The purpose of this paper is to show several applications of the general equations which I studied in my memoir submitted to "Mathematische Annalen" ¹ (see No. 9 of the present publication). For this I give a new solution to the "Waiting problem" dealt with in an extensive memoir by Polyachek. ²

The essence of the problem is as follows. (Naturally, the same mathematical problem can arise when studying other real phenomena, therefore such expressions as “telephone lines” or “conversation” are used here only for the sake of illustration.)

Assume that, in a telephone station, there are $n$ lines over which telephone conversations can take place. At any moment there are $m$ clients that either have a conversation or await their turn; the latter takes place only for $m > n$, and the length of the waiting line is $m - n$. Theoretically, $m$ can take all non-negative integer values:

$$m = 0, 1, 2, \ldots.$$ 

We denote by

$$Q_0(t), Q_1(t), Q_2(t), \ldots, Q_m(t), \ldots$$

the probabilities of these values at time $t$. Clearly,

$$\sum_{m=0}^{\infty} Q_m(t) = 1. \quad (1)$$

If we know $m$ at time $t_0$, then

$$Q_m(t_0) = 1, \quad Q_i(t_0) = 0, \quad i \neq m. \quad (2)$$

The first problem to be solved is to calculate the probabilities $Q_m(t)$ for $t > t_0$ if we know their initial values $Q_m(t_0)$, which, in particular, can take the form (2). The second problem is to determine the expectation and the distribution law for the waiting time for a client arriving at time $t$.

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¹ This article, however, does not suggest that the reader has read this memoir, apart from the proof of several auxiliary statements.

² Math. Z. 32 (1930), 64-100.
§2. We proceed from the following two assumptions on the character of the random events studied.

I. At any infinitely small time interval \((t, t + dt)\) a new client arrives at the station with probability \(\alpha(t)dt\). More precisely, at any interval \((t, t + \Delta)\) the probability of a new client arriving at the station is

\[
\alpha(t)\Delta + o(\Delta),
\]

whereas the probability for more than one client to arrive is

\[
o(\Delta);
\]

here, as usual, \(o(\Delta)\) is an infinitesimal with respect to \(\Delta\); we assume that the function \(\alpha(t)\) is continuous. The probabilities relating to the interval \((t, t + \Delta)\) do not depend on the number of clients arriving earlier or on the times of their arrival.

II. The probability of finishing a conversation maintained by a client at a moment \(t\) during the time interval \((t, t + dt)\) equals \(\beta dt\); it does not depend on the length of the conversation prior to the moment \(t\). This gives us the exponential distribution for the length of a conversation: the probability that this length is between \(t\) and \(t + dt\) is

\[
p(t)dt = \beta e^{-\beta t}dt.
\] (3)

In this case the expected length of a conversation is

\[
D = \int_{0}^{\infty} \beta te^{-\beta t} dt = \frac{1}{\beta}.
\] (4)

Instead of distribution (3) Polyachek assumes that the length of a conversation is always equal to some constant \(D\); this assumption is as arbitrary as ours.

§3. Denote by \(Q_{mp}(t, t + \Delta)\) the conditional probability that at time \(t + \Delta\) there are \(p\) clients at the station if at time \(t\) there were \(m\) clients. The probability that within the interval \((t, t + \Delta)\) more than one client will arrive, or more than one on-going conversation will be finished, is \(o(\Delta)\). Therefore

\[
Q_{mp}(t, t + \Delta) \leq \omega(\Delta) = o(\Delta), \quad |p - m| > 1,
\] (5)