Wald and a number of other American authors have given interesting theorems concerning the sums

\[ \zeta_\nu = \xi_1 + \xi_2 + \ldots + \xi_\nu \]

of the first \( \nu \) random variables from an infinite sequence

\[ \xi_1, \xi_2, \ldots, \xi_n, \ldots \]

where the number \( \nu \) of terms is a random variable (see [1]–[3], where references to earlier literature can be found). In their method of proof these theorems go back to the work of one of the authors of the present paper [4], where for estimating the probability

\[ P\{ \max_{1 \leq n \leq N} |\zeta_n - A_n| \geq h \} \]

he considered sums \( \zeta_\nu \) with index \( \nu \) equal to the first number \( n \) for which

\[ |\zeta_n - A_n| \geq h. \]

The inequality proved in [4] (see also [5], p.154) can easily be derived from Theorem 5 of the present paper.

Further we give very simple proofs for theorems of Wald type relating to the first and second moments. Our conditions for the applicability of basic identities are somewhat broader than those of Wald and Wolfowitz. Our generalization of the conditions for the applicability of these identities is important for certain applications.

In what follows, \( \nu \) denotes a random variable that can take only non-negative integer values, and

\[ n = 0, 1, 2, 3, \ldots \]

The event that \( \nu = n \) will be denoted by

\[ S_n = \{ \nu = n \}, \]

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and the probability of this event by

\[ p_n = P(S_n). \]

Moreover, we set

\[ P_n = P(\nu = n) = \sum_{m=n}^{\infty} p_m. \]

Expectations of random variables,

\[ E(\eta) = \int_U \eta dP \]

will be understood in the sense of the abstract Lebesgue integral over the set of elementary events \( U \). Accordingly, the expectations, when they exist, are always finite, and the existence of \( E(\eta) \) implies the existence of \( E(|\eta|) \). The conditional probability distributions and conditional expectations are understood in the sense explained in [6].

Of basic importance for all theorems of Wald type is the assumption

\[(w) \text{ For } n > m \text{ the random variable } \xi_n \text{ and the event } S_m \text{ are independent.} \]

According to [6], \((w)\) means that for \( n > m \) the conditional distribution of \( \xi_n \) under the condition \( S_m \) coincides with the unconditional distribution

\[ P_{\xi_n}(A|S_m) = P_{\xi_n}(A). \]

**Theorem 1.** If condition \((w)\) holds and the expectations

\[ E(\nu) \text{ and } E(\xi) = a, \quad E(|\xi_n|) = c, \]

exist, where \( a \) and \( c \) do not depend on \( n \), then the expectation of \( \zeta_\nu \) exists and equals

\[ E(\zeta_\nu) = aE(\nu). \quad (1) \]

Since

\[ E(\nu) = \sum_{n=1}^{\infty} p_n n = \sum_{n=1}^{\infty} P_n, \]

Theorem 1 is an obvious consequence of the following more general statement:

**Theorem 2.** If condition \((w)\) holds, the expectations

\[ E(\xi_n) = a_n, \quad E(|\xi_n|) = c_n \]