9. ON ANALYTICAL METHODS IN PROBABILITY THEORY*

The object of investigation

A physical process (a change of a certain physical system) is called stochastically determined if, knowing a state $X_0$ of the system at a certain moment of time $t_0$ we also know the probability distribution for all the states $X$ of this system at the moments $t > t_0$.

I systematically consider the simplest cases of stochastically determined processes, and primarily, processes continuous in time (this is what makes the method essentially new: so far, a stochastic process has usually been considered to be a discrete sequence of separate "events").

If the set $\mathfrak{A}$ of different possible states of the system is finite, then a stochastic process can be characterized using ordinary linear differential equations (Chapter II). If a state of the system depends on one or several continuous parameters, then the corresponding analytic apparatus reduces to partial differential equations of parabolic type (Chapter IV) and we obtain various distribution functions, the normal Laplace distribution being the simplest.

INTRODUCTION

1. In order to subject social or natural phenomena to mathematical treatment, these phenomena should first be schematized. The fact is that mathematical analysis can only be applied to studying changes of a certain system if every possible state of this system can be completely determined using known mathematical techniques, for example, by the values of a certain number of parameters. This mathematically defined system is not a reality itself, but a scheme that can be used to describe reality.

Classical mechanics makes use only of the schemes for which the state $y$ of a system at time $t$ is uniquely determined by its state $x$ at any preceding time $t_0$. Mathematically this can be expressed by the formula

$$y = f(x, t_0, t).$$

If such a unique function $f$ exists, as is always assumed in classical mechanics, then we say that our scheme is a scheme of a purely deterministic

process. These purely deterministic processes also include processes when the state $y$ is not completely determined by giving a state $x$ at a single moment of time $t$, but also essentially depends on the pattern of variation of this state $x$ prior to $t$. However, usually it is preferred to avoid such a dependence on the preceding behaviour of the system, and to do this the notion of the state of the system at time $t$ is generalized by introducing new parameters.\footnote{A well-known example of this method is to introduce, in addition to positions of points, the components of their velocities when describing a state of a certain mechanical system.}

Outside the realm of classical mechanics, along with the schemes of purely deterministic processes, one often considers schemes in which the state $x$ of the system at a certain time $t_0$ only determines a certain probability of a possible event $y$ to occur at a certain subsequent moment $t > t_0$. If for any given $t_0$, $t > t_0$, and $x$ there exists a certain probability distribution for the states $y$, we say that our scheme is a scheme of a stochastically determined process. In the general case this distribution function can be represented in the form

$$P(t_0, x, t, \mathcal{E})$$

where $\mathcal{E}$ denotes a certain set of states $y$, and $P$ is the probability of the fact that at time $t$ one of the states $y$ belonging to this set will be realized. Here we face a complication: in general, this probability cannot be determined for all sets $\mathcal{E}$. A rigorous definition of a stochastically determined process which enables one to avoid this complication is given in §1.

As in the case of a purely deterministic process, we could also have considered here schemes in which the probability $P$ essentially depends not only on the state $x$ but also on the past behaviour of the system. Still, this influence of the past behaviour of the process can be bypassed using the same method as in the scheme of a purely deterministic process.

Note also that the possibility of applying a scheme of either a purely deterministic or a stochastically determined process to the study of some real processes is in no way linked with the question whether this process is deterministic or random.

2. In probability theory one usually considers only schemes according to which any changes of the states of a system are only possible at certain moments...