Chapter 2:

$M$-valued Sets and Sheaves over Integral Commutative
CL-Monoids
§0. Introduction

The concept of Heyting algebra valued sets is intrinsically linked to intuitionistic models, sheaves and topos theory. Precursors of these ideas appear already in D. S. Scott’s and R. Solovay’s work on Boolean-valued models from the mid-sixties [Scott 1967]. Subsequently D. Higgs uses Boolean-valued sets in his unpublished, but widely circulated paper [Higgs 1973] and demonstrates that the category of sheaves over a complete Boolean algebra B is equivalent to the category of B-valued sets and maps in the original Scott-Solovay sense. The concept of Heyting algebra valued sets and its relationship to intuitionistic logic has been extensively studied by M. Fourman and D. S. Scott in their contributions to the Research Symposium on Applications of Sheaf Theory, Durham, NC, 1977 [Fourman and Scott 1979; Scott 1979]. We can summarize the situation as follows: let $\Omega$ be a complete Heyting algebra; then $1^\Omega$-valued sets are intuitionistic models of the formalized mathematical theory of identity with existence predicate, and $2^\Omega$ the category $sh(\Omega)$ of sheaves over $\Omega$ forms a topos. In this context we mention explicitly the fact that the unique classification of subobjects in $sh(\Omega)$ depends essentially on the existence of non-trivial objects with local support.

On the other hand Łukasiewicz logic is the internal logic of the category $\text{MET1}$ of metric spaces and 1-Lipschitz continuous maps [Höhle 1990a]. Disregarding for a moment the initial object, $\text{MET1}$ contains only objects with global support; hence the classification of external subobjects by (pseudo-)characteristic morphisms cannot be unique. In order to improve the classification of extremal subobjects in $\text{MET1}$, the situation in $sh(\Omega)$ suggests an introduction of “metric spaces with local support”. This means syntactically an addition of an existence predicate to the formalized theory of identity or semantically an interpretation of an existence predicate with respect to Łukasiewicz logic.

If we step back for a moment we realize that all semantic interpretations of intuitionistic logic as well as of the Łukasiewicz-Wajsberg axioms by means of truth values are at least based on integral, commutative cl-monoids [Birkhoff 1973]. Hence with regard to non-classical interpretations of local existence we can also ask this question: