Chapter 9

Improperly Posed Problems in Heat Transfer

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Abstract

In this chapter the numerical solution of two inverse Laplace type problems which naturally occur in heat transfer and are improperly posed are investigated. Three different mathematical models, namely direct, least squares and minimal energy methods, are presented for the two problems. The Boundary Element Method is employed and it is found that the minimal energy method always gives a good, stable approximation to the solution, whereas the direct and least squares methods do not.

9.1 Introduction

One may regard a problem as being well posed if a unique solution exists which depends continuously on the data, otherwise it is improperly posed. In order to give a more precise definition one must indicate in what space the solution is to lie, as well as a measure of the continuous dependence. In solving the Laplace equation, which describes a steady heat conductive problem for a physical variable, say $T$, then if either $T$ or $\partial T/\partial n$ is specified at all points on the boundary of a region ($T$ must be specified at least at one point on the boundary), then $T$ can be uniquely determined at all interior points of the region. This class of problems can be solved using either Finite Difference, Finite Element or Boundary Element Methods. Lavrentiev [12] discussed bounded solutions of the Laplace equation in a special two-dimensional domain such that the Cauchy data is continuous. Further, Payne [14],[15] obtained solutions of more general second-order elliptic equations. Whilst Han [8] studied an energy bounded solution of second-order elliptic equations and he proved this solution is dependent on the Cauchy data being continuous. Falk and Monk [5] investigated error estimates for a regularisation method for approximating the Cauchy problem for Poisson’s equation on a rectangle. However, in numerous experimental situations it is not always found to be possible to specify a boundary condition at all points on the boundary of the region. For example, in heat transfer problems many experimental
impediments may arise in measuring or producing given boundary conditions. The physical situation at the boundary may be unsuitable for attaching a sensor or the accuracy of a boundary measurement may be seriously impaired by the presence of the sensor. Frequently it is possible to determine, or specify, either the function $T$ or $\partial T/\partial n$ (i.e. the temperature or the heat flux) on part of the boundary of the region and to be unable to give any information on the remaining part of the boundary. Clearly this is insufficient information in order to determine the function $T$ everywhere within the region of space. Experimentally however, in heat transfer applications, extra sensors may be inserted into the interior region of interest and the temperature measured at these locations, in order to provide more information. The question then arises as to whether, given $T$ or $\partial T/\partial n$ on part of the boundary and $T$ at a number of interior points of the domain, it is possible to determine uniquely the temperature distribution within the region of interest. One of the aims of this chapter is to use the Boundary Element Method (BEM) to obtain the numerical solution to this class of improperly posed problems and we will refer to this as problem I.

In order to illustrate the numerical procedures for solving problem I, all the calculations have been performed in a square region, with either $T$ or $\partial T/\partial n$ given on 3 sides of the square. Further interior information has been given on a straight line. Extension of the work to more irregular shaped boundaries and to the interior information being given at random positions is straightforward and some solutions have been found in these situations. Therefore we let $\Omega \subseteq \mathbb{R}^2$ be a square, such that each side is of unit length, $\partial \Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1$ is one side of the square and $\Gamma_2$ denotes the other three sides. For the purpose of illustrating the solution procedures we let $\Gamma_0 = \{0.25 \leq x \leq 0.75, \; y = y_0\} \subset \Omega$, where $y_0$ is a preassigned value such that $0 < y_0 < 1$, see Fig. 9.1, and on $\Omega$ we consider a steady heat conduction problem

\[
\begin{align*}
\nabla^2 T &= 0 & \text{in } \Omega, \\
T(x,y) &= \phi(x,y) (\text{or } \partial T/\partial n(x,y) = \psi) & (x,y) \in \Gamma_2, \\
T(x,y) &= g(x,y) & (x,y) \in \Gamma_0.
\end{align*}
\] (9.1)

Another problem, II, that we will consider in this chapter is the steady state solution of the nonlinear heat conduction equation,

\[
\nabla \cdot (f(T) \nabla T) = 0 \quad \text{in } \Omega
\] (9.2)

where $f(T)$ is the thermal conductivity of the body which is temperature dependent. We will assume that at every point on the surface of the body either the temperature or the heat flux is prescribed, although it is easy to extend the analysis to include linear combinations of these quantities or even nonlinear boundary conditions, e.g. radiative conditions. If $f(T)$ is known then the techniques as described in Ingham and Kelmanson [10] may be applied. However, frequently in practice the detailed variation of the thermal conductivity with temperature is unknown but extra information in the interior of the body is known, e.g. the temperature may be measured at a number of points within the body. This phenomena also falls into the general class of problems known as improperly posed heat conduction problems since more