Chapter 13

Computer Methods II

Determining the dynamic response of a structure is one of the most demanding challenges for implementing matrix methods on a computer. This divides into two distinct but highly related problems: direct integration in time of the dynamic equilibrium equations, and performing a modal analysis. Reference [5] is an excellent source of additional material.

Direct integration will be conceived as a sequence of pseudo-static problems (one at each time step) with a time varying load that also depends on the inertia properties. Crucial considerations in this type of incremental solution are the questions of accuracy and numerical stability. We first develop a few basic tools that aid in answering these questions, and then analyze two methods of time integration.

The power of modal analysis is that it shows the way for replacing a large dynamic system by one of a much smaller size. Indeed, for many structural dynamics problems, it is usually only the first ten or twenty modes that are of interest. Therefore, emphasis will be given to solving the partial eigenvalue problem; that is, only the lower eigenvalues of a large system will be solved for. We first introduce the concept of repeated orthogonal transformations as a means to reduce a matrix to diagonal form. This is implemented in the Jacobi rotation method. This method, however, is very inefficient for large systems, but the ideas it embodies are incorporated with the vector iteration method to produce the subspace iteration scheme. In this scheme, we can iterate simultaneously on many eigenvectors to give a robust partial solution.

13.1 Finite Differences

When it is inconvenient to integrate a differential equation by analytical means, we usually resort to a step-by-step integration procedure. In this approach, the response is evaluated for a series of short time increments $\Delta t$. The condition of dynamic equilibrium is established at the beginning and end of each interval, and the motion of the system during the time increment is evaluated approximately. The complete response is obtained by using the velocity and displacement computed at the end of one interval as the initial conditions for the next interval.
In the interval, the functions are replaced by simple polynomial representations in terms of just the end values. Such approximations are called finite differences. While the application of this process seems straight-forward, several numerical difficulties arise which can profoundly impact on the quality of the approximate solutions. This section addresses some of these issues before we proceed to the application to our structural systems.

When a differential equation is approximated by a difference equation, there is introduced an error called truncation error; to find the conditions under which the truncation error can be satisfactorily controlled is the problem of convergence. When the difference equation is solved numerically, additional errors due to round-offs are introduced. To find the conditions under which the round-off errors remain sufficiently small as many steps are executed is the problem of stability. Unless the numerical representation of the differential equations is both convergent and stable, the results derived from its use are generally not good approximations to the true values.

## Difference Equations

With reference to Figure 13.1, suppose we want an approximation to the derivative of the function at time \( t \). Assume we know the function \( f(t) \) at discrete times

\[
\ldots, f(t-h), f(t), f(t+h), \ldots
\]

where \( h \) is the spacing between the known values. These discrete values are related to each other through the following Taylor series approximations

\[
f(t + h) \approx f(t) + \frac{df(t)}{dt} h + \ldots \\
f(t - h) \approx f(t) - \frac{df(t)}{dt} h + \ldots
\]

We can now use these expansions to get various approximations for the derivative at time \( t \). From the first we get the forward approximation, the backward approximation from the second, and the combination of the two will give the central difference approximation. That is, respectively,

forward: \( \frac{df(t)}{dt} \approx \frac{1}{h} [f(t + h) - f(t)] \)

backward: \( \frac{df(t)}{dt} \approx \frac{1}{h} [f(t) - f(t - h)] \)

central: \( \frac{df(t)}{dt} \approx \frac{1}{2h} [f(t + h) - f(t - h)] \)

Similarly, if we view the function \( f(t) \) as being the derivative of another function \( f(t) = dg(t)/dt \), then we can construct formulas for the second derivatives as

forward: \( \frac{d^2 g}{dt^2} \approx \frac{1}{h} [g'(t + h) - g'(t)] = \frac{1}{h^2} [g(t + 2h) - 2g(t + h) + g(t)] \)