CHAPTER 5 PERIODIC STEADY STATE

5.1 Fourier series · Excitation and response spectra

Harmonic analysis allows one to calculate the steady state of a dissipative oscillator excited by any periodic external force $f(t)$. We mean, by the steady state, that state which maintains itself after the disappearance of transitory terms. Let us recall that $f(t)$ of period $T = \frac{2\pi}{\omega}$ (figure 5.1), can be decomposed into the Fourier series.

$$f(t) = \frac{1}{2}F_0 + \sum_{n=1}^{\infty} (A_n \cos n\omega t + B_n \sin n\omega t) \quad (5.1)$$

In this expression, the index $n$ varies from one to infinity while the coefficients are given by the integrals

$$\begin{align*}
F_0 &= \frac{2}{T} \int_0^T f(t) \, dt \\
A_n &= \frac{2}{T} \int_0^T f(t) \cos n\omega t \, dt \\
B_n &= \frac{2}{T} \int_0^T f(t) \sin n\omega t \, dt
\end{align*} \quad (5.2)$$

![Fig. 5.1 Periodic external force](image-url)
When the function \( f(t) \) is even \( (f(-t) = f(t)) \), the constants \( B_n \) are zero and the series consists only of cosine terms. Conversely, if \( f(t) \) is odd \( (f(-t) = -f(t)) \), the constants \( A_n \) are zero and the series consists only of sine terms.

By grouping the cosines and the sines for the same angular frequency, \( f(t) \) becomes

\[
f(t) = \frac{1}{2} F_0 + \sum_{n=1}^{\infty} F_n \cos(nwt - \psi_n)
\]

with

\[
\begin{align*}
F_n &= \sqrt{\frac{A_n^2 + B_n^2}{n}} \\
tg \psi_n &= \frac{B_n}{A_n}
\end{align*}
\]

The term \( F_1 \cos(\omega t - \psi_1) \) is the fundamental of the external force; the terms of higher order \( (n > 1) \) are called harmonics.

Let us return now to the equation for the oscillator

\[
m \ddot{x} + c \dot{x} + k x = f(t)
\]

It is a linear differential equation, which allows one to superpose the particular solutions corresponding to each term of (5.3). One will have then

\[
x(t) = \frac{1}{2} F_0 \frac{1}{k} + \sum_{n=1}^{\infty} X_n \cos(nwt - \psi_n - \phi_n)
\]

This result shows that \( x(t) \) is a periodic function, with the same period as \( f(t) \), and so justifies the expression periodic steady state.

The amplitude \( X_n \) of the \( n \)th harmonic can be calculated by means of the relation (4.5)

\[
X_n = \frac{F_n}{\sqrt{(k - n^2 \omega^2 m)^2 + n^2 \omega^2 c^2}} = \mu_n X_{sn}
\]