Chapter 7.
Representations of the Groups \( SU(1,1) \) and \( SL(2, \mathbb{R}) \) in Mixed Bases.
The Hypergeometric Function

7.1. The Realization of Representations \( T_\chi \) in the Space of Functions on the Straight Line

In the preceding chapter we have introduced representations \( T_\chi, \chi = (\tau, \varepsilon) \), of the group \( SU(1,1) \) and have studied their matrix elements in the basis \( \{ e^{in\theta} \} \) which diagonalizes the operators \( T_\chi(g(t)), g(t) = \text{diag}(e^{it/2}, e^{-it/2}). \) Now we study other realizations of these representations. It will be convenient for us to consider representations \( T_\chi \) of the group \( SL(2, \mathbb{R}) \) which is isomorphic to \( SU(1,1) \). Subgroups and decompositions, considered below, have simpler form for \( SL(2, \mathbb{R}) \).

7.1.1. Parametrization of \( SL(2, \mathbb{R}) \). The group \( SL(2, \mathbb{R}) \) consists of real matrices \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \), \( \alpha \delta - \beta \gamma = 1 \). Let us introduce on \( SL(2, \mathbb{R}) \) the parametrization, closely connected with that of the group \( SU(2) \) by Euler angles. It is based on the following statement.

**Theorem 1.** Any matrix \( g \) of \( SL(2, \mathbb{R}) \), with all its elements being non-zero, can be represented in the form

\[
g = d_1(-e)^{\varepsilon_1}s^{\varepsilon_2}pd_2, \tag{1}
\]

where \( \varepsilon_1, \varepsilon_2 = 0 \) or \( 1 \); \( d_1 = \text{diag}(e^{\varphi/2}, e^{-\varphi/2}), d_2 = \text{diag}(e^{\psi/2}, e^{-\psi/2}); \)

\[
-e = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

and \( p \) is a matrix of one of the types

\[
p = \begin{pmatrix} \cosh \frac{\theta}{2} & \sinh \frac{\theta}{2} \\ \sinh \frac{\theta}{2} & \cosh \frac{\theta}{2} \end{pmatrix}, \quad -\infty < \theta < \infty, \tag{2}
\]

\[
p = \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \tag{3}
\]

At first we prove the following lemma.

**Lemma.** Let \( g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) be a matrix of \( SL(2, \mathbb{R}) \) such that \( |\alpha| = |\delta|, |\beta| = |\gamma|, |\alpha| \geq |\beta| \) and \( \alpha > 0 \). Then it is of the form (2) or (3).

**Proof.** Since \( \alpha \delta - \beta \gamma = 1 \) and \( |\alpha \delta| \geq |\beta \gamma| \), then \( \alpha \delta > 0 \). Hence, \( \alpha = \delta > 0 \). Let \( \alpha = \delta \geq 1 \). Since \( \alpha \delta - 1 = \beta \gamma \), then \( \beta \gamma \geq 0 \). Because of \( |\beta| = |\gamma| \) we have \( \beta = \gamma \).
Setting $\beta = \sinh \frac{\theta}{2}$, we obtain $\alpha = \cosh \frac{\theta}{2}$. Consequently, $g$ is of the form (2). Now let $\alpha = \delta < 1$. The equality $\alpha \delta - 1 = \beta \gamma$ implies $\beta \gamma < 0$ and, therefore, $\gamma = -\beta$.

For $\alpha = \cos \frac{\theta}{2}$, $\beta = \sin \frac{\theta}{2}$ the matrix $g$ is of the form (3). Since $\alpha > 0$, $|\alpha| > |\beta|$, then here $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Now we prove the theorem. We set $e^\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $e^\psi = \begin{pmatrix} \alpha \gamma & \beta \delta \\ \alpha \delta & \beta \gamma \end{pmatrix}$ and denote by $g_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}$ the matrix $d_1^{-1}g d_2^{-1}$, where $d_1 = \text{diag}(e^{\varphi/2}, e^{-\varphi/2})$, $d_2 = \text{diag}(e^{\psi/2}, e^{-\psi/2})$.

Due to the choice of $\varphi$ and $\psi$, the equalities $|\alpha_1| = |\delta_1|$ and $|\beta_1| = |\gamma_1|$ are satisfied.

Let us denote by $p$ the matrix $p = s^{-\varepsilon_2}(-e)^{-\varepsilon_1} g_1$, $p = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}$, where

a) $\varepsilon_1 = \varepsilon_2 = 0$ if $|\alpha_1| \geq |\beta_1|$, $|\varepsilon_1| > 0$,

b) $\varepsilon_1 = 1, \varepsilon_2 = 0$ if $|\alpha_1| \geq |\beta_1|$, $|\varepsilon_1| < 0$,

c) $\varepsilon_1 = 0, \varepsilon_2 = 1$ if $|\alpha_1| < |\beta_1|$, $|\varepsilon_2| > 0$,

d) $\varepsilon_1 = \varepsilon_2 = 1$ if $|\alpha_1| < |\beta_1|$, $|\varepsilon_1| < 0$.

Then the inequalities $|\alpha_2| = |\delta_2| > |\beta_2| = |\gamma_2|$, $\alpha_2 > 0$ hold. Consequently, the matrix $p$ is of the form (2) or (3). But then $g = d_1 g_1 d_2 = d_1 (-e)^{\varepsilon_1} s^{\varepsilon_2} p d_2$. The theorem is proved.

It follows from the theorem that each matrix $g$ of $SL(2, \mathbb{R})$ is given by numbers $\varphi, \theta, \psi$ and numbers $\varepsilon_1, \varepsilon_2$ taking the values 0 and 1. Besides, it has to be indicated whether the matrix $p$ is of the form (2) or (3). Thus, we obtain eight domains in $SL(2, \mathbb{R})$, characterized by the values of $\varepsilon_1, \varepsilon_2$ and by the type of the matrix $p$. In each of these domains the matrix $g$ is uniquely determined by $\varphi, \theta, \psi$. Moreover, if $g$ is of the form (2), then $\theta$ varies from $-\infty$ to $\infty$, and if $g$ is of the form (3), then $\theta$ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Now we consider matrices $g$ of $SL(2, \mathbb{R})$ for which one of the matrix elements is equal to zero. As in Theorem 1, we can see that $g$ is represented in the form

$$g = d(-e)^{\varepsilon_1} s^{\varepsilon_2} p s \varepsilon_3,$$

where $d = \text{diag}(e^{\varphi/2}, e^{-\varphi/2})$, $p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\varepsilon_1, \varepsilon_2$ and $\varepsilon_3$ take the values 0 and 1, and $-e$, $s$ are the same as in (1). For example, if $g = \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}$, $\gamma > 0$, then $g = d p s$, where $e^{-\varphi/2} = -\gamma$ and $x = -\frac{\delta}{\gamma}$.

7.1.2. A new realization of representations $T_\chi$. The representations $T_\chi, \chi = (\tau, \varepsilon)$, of the group $SU(1, 1)$, described in Section 6.4.1, are representations of the group $SL(2, \mathbb{R})$, isomorphic to $SU(1, 1)$ (see Section 6.1.3). To obtain a convenient expression for these representations, we note that the isomorphism of Section 6.1.3 transfers the representation

$$(T_\chi(h)\Phi)(z) = \Phi(az + \bar{b}z), \quad h = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix},$$