AN APPLICATION OF ZECKENDORF'S THEOREM

Roger V. Jean

INTRODUCTION

The author is involved in research in biomathematics, more precisely in the application of mathematics to plant biology. From the very beginning he was confronted with Fibonacci numbers. Indeed these numbers arise in what is considered to be the bugbear of botany. Their overwhelming presence in the secondary spirals on plants has puzzled many research workers. Many theories and models have been elaborated but the problem is still unsolved. The author produced dozens of articles and the book [2] mentioned in the references on the subject. This book has been reviewed in many places especially in [1]. Another book by the same author on the same subject is in preparation.

As tools for teaching, the author produced articles that may be particularly attractive for Fibonacci's disciples. Some of them are listed in the references ([3, 4, 5]).

Fibonacci numbers are not only involved in advanced mathematics and research subjects, but also in recreational amusements and tricks. These tricks often apply important mathematical results to which the attention is happily attracted. The paper presents such a trick, based on an application of Zeckendorf's theorem on the completeness of the Fibonacci sequence.

A MATHEMATICAL MYSTIFICATION WITH CARDS

Consider the nine sets of integers from 1 to 75, given at the end of the paper. You will notice that any given number can appear in several of the sets. Each of the sets are on separate
cards which can be cut out and shuffled before making the trick, to make it more mysterious.
The idea is to ask someone to choose a number between 1 and 75, and to guess that number
when he or she finished telling you on which cards is the number he or she chose. The only
prerequisite to make the trick a convincing demonstration of mental power and mental reading
is to know how to add integers. The theorem (see below) at the basis of the trick is not so
simple however, but its knowledge is not required for the performance.

Let us take an example. Suppose that 19 was chosen. You are then told that this
number is on cards 1, 4, and 6. You announce immediately that the number is precisely 19 by
summing up the first numbers in each set, here 1 + 5 + 13. The numbers in the sum are all
Fibonacci numbers of course, as is the first number on every card. Given the theme of the
Conference you have certainly noticed that peculiarity of the cards, but as we move away from
Fibonacci's circles the trick will certainly appear to be more fantastic. Try again: you are told
that the number chosen is on cards 3, 5, and 8. The sum to perform is 3 + 8 + 34, and the
number is 45.

**DISSOLVING THE MYSTERY - THE CONSTRUCTION RULE**

How can this trick be explained? Zeckendorf's theorem states that every positive integer
can be decomposed in a unique way as the sum of distinct Fibonacci numbers, such that no two
such numbers are consecutive in the Fibonacci sequence. Because of Zeckendorf's theorem there
is a unique way to decompose an integer and thus to put it on the cards, and we say that the
Fibonacci sequence is complete. For each Fibonacci number there is a card containing integers,
and each card starts with a different Fibonacci number. If the trick had been made for the first
100 integers we would thus have needed 10 cards (one more starting with 89, the Fibonacci
number following 55 on card 9).

The theorem tells us how to build the cards. For example 40 = 34 + 5 + 1 and 77 =
55 + 21 + 1. Always start the decomposition of the given number with the largest included
Fibonacci number, then the next largest and so on. It follows that 40 must be on cards 1, 4,
and 8 (given that 1, 5, and 34 are the first numbers on these cards), while 77 must appear on
cards 1, 7, and 9 (assuming in that case that the trick has been extended over 75).

**CREATING MORE MYSTERIES**

Now that you know how the cards are made you can make the trick seemingly more
difficult by extending it to the numbers between 1 and 100. For example 99 would be on the