THE NEHARI PROBLEM

Over the last years, the Nehari problem has been receiving considerable attention. This interest is because many control problems including $H^\infty$ optimization, robustness with respect to dynamic modeling uncertainty, disturbance attenuation, and mixed sensitivity may be reduced to Nehari problems.

Different approaches have been proposed for solving this problem; we mention among them those using interpolation techniques, $\gamma$-iteration approaches, all-pass embedding methods, and, more recently, approaches based on the generalized Popov–Yakubovich theory; corresponding bibliographical details are given in Notes and References.

This chapter is concerned with one and two-block Nehari problems, whose suboptimal solutions are obtained using the generalized Popov–Yakubovich theory; in addition, optimal solutions are also determined via the singular perturbations approach. Illustrative examples including a model matching problem and an optimal robust design with respect to the left coprime factorization are also presented.

Although the two-block Nehari problem is more complex than the one-block case, we shall treat first this problem which will allow us to obtain directly the solution corresponding to the one-block Nehari problem.

4.1. THE TWO-BLOCK CASE

4.1.1. SUBOPTIMAL SOLUTION

Consider the system having the following structure

$$ T := \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C_1 & D_1 \\ C_2 & D_2 \end{bmatrix}, $$

(4.1)

where $A \in \mathbb{R}^{n \times n}$ is antistable, $B \in \mathbb{R}^{n \times m}$, $C_i \in \mathbb{R}^{p_i \times n}$, $D_i \in \mathbb{R}^{p_i \times m}$, $i = 1, 2$. Then, given $\gamma > 0$, the suboptimal two-block Nehari problem consists in finding a stable system $S$ such that

$$ \left\| \begin{bmatrix} T_1 + S \\ T_2 \end{bmatrix} \right\| \infty < \gamma. $$

(4.2)
In the following we shall assume that \( D_1 = 0 \); there is no loss of generality in this assumption since if \( S \) is a solution to the Nehari problem with \( D_1 = 0 \), then \( S - D_1 \) is a solution to the Nehari problem with arbitrary \( D_1 \).

Necessary and sufficient conditions for solvability of the suboptimal two-block Nehari problem, as well as a solution of this problem, are given by the following result:

**Theorem 4.1** The following assertions are true:

1. The suboptimal two-block Nehari problem (4.2) has a solution if and only if
   \[
   \gamma^2 I - D_2^T D_2 > 0, \tag{4.3}
   \]
   and the ARE
   \[
   -AZ - ZA^T + \left[ Z(C_2^T D_2 - XB) - B \right] \left( \gamma^2 I - D_2^T D_2 \right)^{-1} \times \left[ (D_2^T C_2 - B^T X)Z - B^T \right] - ZC_1^T C_1 Z = 0 \tag{4.3}
   \]
   has a stabilizing positive semi-definite solution \( Z \), where \( X \geq 0 \) is the solution to the Lyapunov equation
   \[
   A^T X + XA - C_1^T C_1 - C_2^T C_2 = 0. \tag{4.4}
   \]
2. If the conditions from point (1) hold, then a solution to the suboptimal two-block Nehari problem is given by
   \[
   S = \begin{bmatrix}
   -\left( A + ZC_1^T C_1 \right)^T & C_2^T D_2 - XB \\
   -C_1 Z & 0
   \end{bmatrix}. \tag{4.5}
   \]

**Proof.** (1) **Necessity** Let \( S \) be a stable system satisfying (4.2); then the condition \( \gamma^2 I - D_2^T D_2 > 0 \) follows directly from

\[
\| T_2 \|_\infty \leq \left\| \begin{bmatrix} T_1 + S \\ T_2 \end{bmatrix} \right\|_\infty < \gamma.
\]

Now we prove that (4.3) has a stabilizing positive semi-definite solution. From (4.2) it follows that

\[
(T_1 + S)^*(T_1 + S) + T_2^* T_2 < \gamma^2 I,
\]

which may be rewritten in the equivalent form

\[
\begin{bmatrix} I & S^* \end{bmatrix} \Pi \begin{bmatrix} I \\ S \end{bmatrix} < 0 \quad \text{on} \quad j\overline{\mathbb{R}}, \tag{4.6}
\]