SOME PROBABILISTIC ASPECTS OF THE ZECKENDORF DECOMPOSITION OF INTEGERS

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1. INTRODUCTION

That any positive integer \( N \) can be represented as a sum of distinct nonconsecutive Fibonacci numbers \( F_i \) is a well-known fact. Apart from the equivalent use of \( F_2 \) instead of \( F_1 \), such a representation is unique [1] and is commonly refereed to as the Zeckendorf Decomposition (or Representation) of \( N \) [ZD(N), in brief].

Let \( N \) be an integer belonging to the interval \([1, F_n - 1]\), with \( n \geq 3 \). Following the symbolism of [6], the ZD(N) is

\[
N = \sum_{i=2}^{n-1} \varepsilon_i F_i \text{ with } \varepsilon_i \in \{0, 1\}, \text{ and } \varepsilon_i \varepsilon_{i+1} = 0,
\]

whereas, following the symbolism of [7], the number \( f(N) \) of non-zero summands in (1.1) is given by

\[
f(N) = \sum_{i=2}^{n-1} \varepsilon_i.
\]

Observe (see (1) of [2]) that the quantity \( f(N) \) is subject to the inequalities

\[
1 \leq f(N) \leq \lfloor (n-1)/2 \rfloor
\]

where the symbol \( \lfloor \cdot \rfloor \) denotes the greatest integer function.

This paper aims to investigate some probabilistic aspects of the ZD(N). To the best of our knowledge, apart from some simple results established in [2], no attempt has been formerly
made to investigate these aspects, so we hope that the results established by us will be of some interest to the readers.

2. A BASIC QUESTION

In our eyes, a basic question concerning the probabilistic aspects of $ZD(N)$ is as follows.

**Question 1:** If $N$ is randomly chosen within the interval $[1, F_n - 1]$ (with $n \geq 3$), what is the probability $P(k, n)$ that $\varepsilon_k = 1$? In other words, what is the probability that the Fibonacci number $F_k (2 \leq k \leq n - 1)$ appears in the $ZD(N)$?

2.1. Answering Question 1

To answer Question 1, let us consider (see Fig. 1) the binary string having the symbol 1 in the position $k$, and the symbol 0 in the positions $k - 1$ and $k + 1$.

\[
\begin{array}{cccccccc}
  & \underbrace{0} & \underbrace{1} & \underbrace{0} & \ & \ & \ & \ \\
\hline
k-3 & \ & \ & \ & k & k+2 & \ & \ \\
\hline
2 & \cdots & k-2 & k & k+2 & \cdots & n-1 & \\
\end{array}
\]

Fig. 1 - $ZD(N)$ with $N \in [1, F_n - 1]$ and $\varepsilon_k = 1$.

The number of integers $N$ having $F_k$ in the $ZD(N)$ clearly equals the number of ways in which the two sub-strings of length $k - 3$ and $n - k - 2$ can be filled with $0, 1, 2, \ldots$ nonadjacent ones. After recalling (see Theorem 2 of [2]) that the number $B_{M,m}$ of distinct binary sequences of length $M \geq 1$ containing $m$ nonadjacent ones and $M - m$ zeros is given by

\[ B_{M,m} = \binom{M - m + 1}{m} \quad (0 \leq m \leq \lfloor (M + 1)/2 \rfloor), \tag{2.1} \]

we are in a position to answer Question 1. In fact, after replacing $M$ by $k - 3$ and $n - k - 2$ in (2.1), we can write

\[
P(k, n) = \frac{1}{F_n - 1} \left( \sum_{m=0}^{\lfloor (k - 2)/2 \rfloor} B_{k - 3, m} \right) \left( \sum_{m=0}^{\lfloor (n - k - 1)/2 \rfloor} B_{n - k - 2, m} \right)
\]

\[
= \frac{1}{F_n - 1} \left( \sum_{m=0}^{\lfloor (k - 2)/2 \rfloor} \binom{k - 2 - m}{m} \right) \left( \sum_{m=0}^{\lfloor (n - k - 1)/2 \rfloor} \binom{n - k - 1 - m}{m} \right) \tag{2.2}
\]

\[
= \frac{F_k - 1 \cdot F_{n - k}}{F_n - 1}. \tag{2.3}
\]