DIRAC OPERATORS AND CLIFFORD GEOMETRY - NEW UNIFYING PRINCIPLES IN PARTICLE PHYSICS?

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Abstract. In this lecture I will report on some recent progress in understanding the relation of Dirac operators on Clifford modules over an even-dimensional closed Riemannian manifold \( M \) and (euclidean) Einstein-Yang-Mills-Higgs models.

Although being a gauge theory, it is well-known that the classical theory of gravity as enunciated by Einstein stands apart from the non-abelian gauge field theory of Yang, Mills and Higgs, which encompasses the three other fundamental forces: the electromagnetic, weak and the strong interaction. General relativity is governed by a variational principle associated with the Lagrangian

\[
S_{GR} = \frac{1}{16\pi G} \int_M * r_M
\]

where \( G \) denotes Newton's constant, \( * \) is the Hodge star and \( r_M \) is the scalar curvature of the space-time \( M \), a closed four-dimensional pseudo-Riemannian manifold of signature \((-+, \ldots +)\). To describe a Yang-Mills-Higgs model the space-time manifold \( M \), apart from the metric, is endowed with additional structure (cf. [10]): Over \( M \) we have both a principal bundle \( P_G \) with structure group, a compact Lie group \( G \) and a Clifford module \( \mathcal{E} \) that is assumed to furnish a representation \( \rho: \mathcal{G} \to \text{Aut}_{C(M)} \mathcal{E} \) of the group of gauge transformations \( \mathcal{G} \) of \( P_G \). Here \( \text{Aut}_{C(M)} \mathcal{E} \) are those automorphisms of \( \mathcal{E} \) which commute with the Clifford action. Let \( A \) be a connection on \( P_G \) with curvature \( R \in \Omega^2(M, \text{ad}(P_G)) \), \( \nabla^\mathcal{E}: \Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathcal{E}) \) be the

\(^1\) Here \( \text{ad}(P_G) \) denotes the vector bundle with fibre the Lie algebra \( LG \) associated to \( P_G \) with respect to the adjoint representation \( Ad: G \to LG \).

associated Clifford connection and $\varphi \in \Gamma(W)$ a section of an additional vector bundle $W$ associated to $P_G$ with induced connection $\nabla^W$. Then a corresponding Yang-Mills-Higgs model is based on the Lagrangian

$$S_{YMH} = -\frac{1}{g^2} \int_M tr(R \wedge *R) + \int_M \left( (\nabla^W \varphi \wedge *\nabla^W \varphi) + *V(\varphi) \right) + \int_M *(\psi, D_\varphi \psi)$$  \hspace{1cm} (2)

where $V: W \to G$ is an invariant quartic polynomial, $\psi \in \Gamma(\mathcal{E})$ and $D_\varphi: \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ denotes a Dirac-Yukawa operator associated to $(\nabla^\mathcal{E}, \varphi)$. The constant $g$ in (2) parametrizes the fibre metric $tr: ad(P_G) \times ad(P_G) \to G$ which is induced by the Killing form on the Lie algebra $LG$ of $G$ and is called the Yang-Mills coupling constant. If $G$ is not simple, it is possible to generalize (2) introducing a separate coupling constant for each simple factor.

With exception of the ‘pure’ Yang-Mills term $S_{YM} = -\frac{1}{g^2} \int_M R \wedge *R$, from a mathematical point of view the lagrangian (2) looks highly artificial. So we briefly comment on its physical significance:

- The bosonic part of a Yang-Mills-Higgs model which describes (non-abelian) gauge forces, is defined by the first two terms of (2). The (covariant) Klein-Gordon lagrangian $S_{KG} = \int_M \nabla^W \varphi \wedge *\nabla^W \varphi$ and the Higgs potential $S_\varphi = \int_M *V(\varphi)$ for the Higgs field $\varphi \in \Gamma(W)$ are added to the pure Yang-Mills part such that the gauge bosons or connections $A^i \in \Omega^1(M, ad(P_G))$ acquire masses. In the theory of electroweak interaction for example, where we have $G_{ew} = SU(2) \times U(1)$ and $V(\varphi) := \frac{\lambda}{4}(\varphi, \varphi)^2 - \frac{\mu^2}{2}(\varphi, \varphi)$ with $\varphi \in \Gamma(M \times \mathbb{C}^2)$, $(\cdot, \cdot)$ is the standard inner product on $\mathbb{C}^2$ and $\lambda, \mu > 0$.
- The fermionic part $\int_M *(\psi, D_\varphi \psi)\mathcal{E}$ which describes matter fields \footnote{For example, we have leptons and quarks in the Standard Model.} is defined by the Dirac-Yukawa operator $D_\varphi := D + c_Y \phi(\varphi)$. Here $D$ is the Dirac operator corresponding to the Clifford connection $\nabla^\mathcal{E}$ and $\phi: W \to \text{End}_{C(M)}^\mathcal{E}$ is a linear map. The Yukawa coupling $c_Y \in \mathbb{R}$ gives rise to the fermion mass as soon as there exists a non-vanishing $\varphi_0 \in \Gamma(W)$ which minimizes the Higgs-potential $V$ but is only invariant under a subgroup $\mathcal{H} \subset \mathcal{G}$ of the group of gauge transformations.