1. Introduction: exact statistical self-affinity

It is already almost two decades since fractals [1] obtained their name and were developed from merely curious mathematical constructions into an important instrumentation for practically all natural and engineering sciences [2, 3, 4]. These advances were brought about by the efforts to model and characterize complex structures and processes which either cannot be approximated using Euclidean objects or such an approximation proves inefficient. To facilitate the discussion, consider an example taken from the remote sensing practice, see Fig. 1, where a laser profiler measured profile of cultivated soil [5, 6] is shown.

Suppose we attempt to construct, say, a piecewise linear approximation of these data around a grid of argument values. Since the graph clearly depicts a function with very large derivatives, the ranges of validity of the local linear approximations would be extremely narrow. Hence, we would need a very fine grid, a lot of knot points and therefore intolerably many parameters to describe the data. Experience shows that the processes and morphologies encountered in the nature, are usually so irregularly shaped that using a higher order polynomial approximations does not reduce the number of the free parameters significantly. Of course, one can legitimately argue that the only sensible way to deal with such functions is to consider them as realizations of a random process and employ statistical methods for their characterization. Since the natural morphologies are formed under the influence of many independent processes, it is usually correct to assume
that the probability distribution is close to Gaussian. Hence, an adequate description of the structures and processes of interest requires a modeling of their second order statistical moments: variances, autocovariance functions (ACFs) or equivalently the power spectra. In Fig. 1 we show by the solid line the mean of the soil profile \( \langle h \rangle = 102.735 \text{ mm} \) (from the profiler level). Apparently, this value tells little about the actual variability of the structure. Likewise, the standard (sample) deviation from the mean \( \langle h^2 \rangle^{1/2} = 13.895 \text{ mm} \), shown in Fig. 1 by the two dashed lines, provides information of the degree of profile variability but says nothing about the specific form, shape, the presence of variety of different scale peaks, etc. Clearly, an accurate statistical characterization of the soil roughness requires a detailed models of its ACF/power spectrum which at first may appear to be complicated and involving large number of parameters.

The way to simplify such problems seems to be suggested by the nature itself. It is based on the statistical self-affine (self-similar) symmetry [1, 2], which by reasons not entirely understood is ubiquitous in nature. (See for example, [7] for recent ideas to explain this ubiquity.) The statistical self-affinity is demonstrated in the inset of Fig. 1 where portion of the soil profile (distances between 590 and 680 mm) is magnified. It is seen

\[\begin{align*}
\text{soil height profile [mm]} \\
\text{distance [mm]}
\end{align*}\]

*Figure 1.* Illustration of the self-affinity of a soil profile. The inset shows an enlarged view of the marked segment.