Having looked at the basic methods of differentiation and integration in Chapters 4 and 5, we now come to the consideration of differential equations. Initially we shall give some definitions and simple examples. Following this, consideration will be given to some geological examples. As many of the examples chosen are complicated, it is not planned to look at their complete mathematical derivation or proof, but rather to look at the geological implications of their use. It is hoped that the references given will be adequate for those who wish to explore these particular applications further. The use of differential equations in geology is probably one of the most important topics in mathematical geology, and this is likely to continue to be so in the future.

8.1 Definitions and examples

An ordinary differential equation is any relationship between the variables $x$ and $y$, and the derivatives $dy/dx$, $d^2y/dx^2$, etc. The term 'ordinary' serves to distinguish them from partial differential equations, which involve partial derivatives. The order of a differential equation is that of the highest-order derivative occurring, so

\[
\frac{dy}{dx} = f(x) \quad \text{is a first-order differential equation}
\]

\[
\frac{d^2y}{dx^2} + \frac{dy}{dx} = f(x) \quad \text{is a second-order differential equation}
\]

The solution of a differential equation is an equation relating the variables involved which does not contain any differential coefficients. In other words, we integrate to eliminate the differentials:

\[
\frac{dy}{dx} = 3x
\]

\[
f(x) = \int 3x \, dx = \frac{3}{2}x^2 + C
\]
Note that here we include the constant of integration, which leads to the \textit{general solution}. Thus, the equation gives rise to a family of curves, which will have the same slope for a particular value of $x$, as shown in Figure 8.1. If a \textit{particular solution} is required, then an $x$- and corresponding $y$-value must be given. These are referred to as \textit{boundary conditions}, and enable the constant of integration to be calculated. Thus, in the example we have just considered, if $x = 0$ when $y = 2$, then

$$y = \frac{3}{2}x^2 + 2$$

We have already seen that if the equation is in the form

$$\frac{dy}{dx} = f(x)$$

then we integrate directly, so

$$y = \int f(x)\,dx$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{image1.png}
\caption{The family of curves of the differential equation $\frac{dy}{dx} = 3x$, i.e. $f(x) = \frac{3}{2}x^2 + c$ for $c = -1, 3$ and 10.}
\end{figure}